Journal of Functional Analysis 270 (2016) 4213–4255

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Blowup profile for a complex valued semilinear heat equation

Junichi Harada

Faculty of Education and Human Studies, Akita University, 1-1 Tegata-Gakuenmachi, Akita 010-8502, Japan

A R T I C L E I N F O

Article history:
Received 22 February 2015
Accepted 25 March 2016
Available online 5 April 2016
Communicated by B. Chow

Keywords:
Blow-up profile
Simultaneous blow-up

A B S T R A C T

This paper is concerned with finite blow-up solutions of a one dimensional complex-valued semilinear heat equation. We classify blow-up solutions and derive their blow-up profiles under some assumptions. In particular, we discuss the possibility of a nonsimultaneous blow-up.

1. Introduction

We study blow-up solutions of a one dimensional complex-valued semilinear heat equation.

$$z_t = z_{xx} + z^2, \tag{1}$$

where $z(x,t)$ is a complex valued function and $x \in \mathbb{R}$. If $z(x,t)$ is written by $z = a + ib$ $(a, b \in \mathbb{R})$, then (a, b) satisfies

$$a_t = a_{xx} + a^2 - b^2, \qquad b_t = b_{xx} + 2ab.$$

E-mail address: harada-j@math.akita-u.ac.jp.

http://dx.doi.org/10.1016/j.jfa.2016.03.015

This equation is a special case of Constantin–Lax–Majda equation with a viscosity term, which is a one dimensional model for the 3D Navier–Stokes equations (see [4,18–20,10]). The Cauchy problem (1) admits a unique local solution in $L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. We call a solution z *blow-up* in a finite time, if there exists $T > 0$ such that

$$\limsup_{t\to T} \|z(t)\|_{L^\infty(\mathbb{R})} = \limsup_{t\to T} \sqrt{\|a(t)\|^2_{L^\infty(\mathbb{R})} + \|b(t)\|^2_{L^\infty(\mathbb{R})}} = \infty.$$

Moreover we call a point $x_0 \in \mathbb{R}$ a *blow-up point*, if there exists a sequence $\{(x_j, t_j)\}_{j\in\mathbb{N}} \subset \mathbb{R}\times(0,T)$ such that $x_j \to x_0$, $t_j \to T$ and $|z(x_j, t_j)| \to \infty$ as $j \to \infty$. When z is real-valued (i.e. $b \equiv 0$), (1) coincides with a single semilinear heat equation [7]:

$$a_t = a_{xx} + a^2.$$

Blow-up problems for this single equation are well-understood. As for a parabolic system case, blow-up problems are extensively studied in various directions. The asymptotic behavior of blow-up solutions for a parabolic system with a gradient structure is derived by Filippas–Merle [6]. For a parabolic system with no gradient structure case, a blow-up solution for $u_t = \Delta u + v^p$, $v_t = \Delta v + u^q$ is studied by Andreucci–Herrero–Velazquez [2]. More general nonlinear parabolic system with no gradient structure is treated in [3,15, 21,23,24].

We first consider an ODE solution $(a(x,t), b(x,t)) = (a(t), b(t))$ of (1). Then (1) is reduced to

$$a_t = a^2 - b^2, \qquad b_t = 2ab.$$

This ODE system has a unique solution given by

$$a(t) = \frac{T_1 - t}{(T_1 - t)^2 + T_2^2}, \qquad b(t) = \frac{T_2}{(T_1 - t)^2 + T_2^2},$$

where $T_1 = a_0/(a_0^2 + b_0^2)$ and $T_2 = b_0/(a_0^2 + b_0^2)$. Therefore this ODE solution exists globally in time, if $b_0 \neq 0$. From this observation, we expect that an imaginary component b prevents a blow-up in (1). In fact, Guo–Ninomiya–Shimojo–Yanagida [10] prove the following result.

Theorem 1.1. *(See Theorem 1.1 [10].) Suppose that the initial data $(a_0, b_0) \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ satisfy*

$$a_0(x) < Ab_0(x) \qquad \text{for all } x \in \mathbb{R}$$

with some constant $A \in \mathbb{R}$. Then the solution of (1) exists globally in time and $\lim_{t\to\infty} (a(t), b(t)) = (0, 0)$ in $L^\infty(\mathbb{R})$.