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Compact truncated Toeplitz operators ☆

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ABSTRACT

A necessary and sufficient condition is found for a truncated Toeplitz operator with bounded symbol to be compact on the model space.

Let $\mathbb{D}$ be the open unit disk in the complex plane and L^2 denote the Lebesgue space of square integrable functions on the unit circle $\mathbb{T}$. The Hardy space H^2 is the subspace of analytic functions on $\mathbb{D}$ whose Taylor coefficients are square summable. Then it can be also identified (via radial limits) with the subspace of L^2 of functions whose negative Fourier coefficients vanish. Truncated Toeplitz operators are compressions of

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the standard Toeplitz operators on the Hardy space H^2 to its coinvariant subspace, the so-called model space

$$K_\theta = H^2 \ominus \theta H^2,$$

where θ is a non-constant inner function. Let P_θ denote the orthogonal projection from L^2 onto the subspace K_θ. For $\varphi \in L^2$, the truncated Toeplitz operator A_φ with symbol φ is defined by

$$A_\varphi f = P_\theta(\varphi f),$$

on the dense subset $K_\theta \cap L^\infty$ of the space K_θ. The symbol φ is not unique [17].

Although the truncated Toeplitz operators share many fundamental properties of classical Toeplitz operators on the Hardy space, they differ in many crucial ways. For example, compact Toeplitz operators on Hardy space are zero, but there are many nonzero compact truncated Toeplitz operators. For $\varphi \in H^\infty$, Sarason showed that the truncated Toeplitz operator A_φ is compact if and only if $\bar{\theta}\varphi \in H^\infty + C$ (Section VIII.3, [16]). Bessonov [3] showed the Sarason result holds even for $\varphi \in H^\infty + C$. More results on compact truncated Toeplitz operators are contained in [4,15]. Naturally one may ask the following question:

Question. When is a truncated Toeplitz operator A_φ compact on K_θ for $\varphi \in L^\infty$?

In this paper we will answer the question completely via "localization" on support sets. The following is our main result that gives a complete description of the compactness of truncated Toeplitz operators with bounded symbol in terms of function theoretic properties of the symbol and the inner function.

Theorem 1. *Let φ be in L^∞ and θ be a non-constant inner function. The truncated Toeplitz operator A_φ is compact on K_θ if and only if for each m in $M(H^\infty + C)$ one of the following holds.*
(a) $\theta|_{S_m}$ is constant,
(b) $\varphi|_{S_m} \in \theta|_{S_m} H^2(m) + \bar{\theta}|_{S_m}\overline{H^2(m)}$.

Some notations in the above theorem will be introduced in Section 1. One of our motivations is Sarason's characterization of zero truncated Toeplitz operators [17]:

Theorem 2. *If φ is in L^2 and θ is an non-constant inner function, $A_\varphi = 0$ if and only if $\varphi \in \theta H^2 + \overline{\theta H^2}$.*

Roughly speaking, Theorem 1 tells us that the condition in the Sarason theorem (Theorem 2) holds on each support set on which θ is not constant if and only if the truncated Toeplitz operator is compact.

Another motivation is the Axler–Chang–Sarason–Volberg theorem on the compact semicommutator of two Toeplitz operators on the Hardy space [1,20]: