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L^∞ estimates in optimal mass transportation

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We show that in any complete metric space the probability measures μ with compact and connected support are the ones having the property that the optimal transportation distance to any other probability measure ν living on the support of μ is bounded below by a positive function of the L^∞ transportation distance between μ and ν. The function giving the lower bound depends only on the lower bound of the μ-measures of balls centered at the support of μ and on the cost function used in the optimal transport. We obtain an essentially sharp form of this function.

In the case of strictly convex cost functions we show that a similar estimate holds on the level of optimal transport plans if and only if the support of μ is compact and sufficiently close to being a geodesic metric space in the quantitative sense that between any two points there exists a sequence along which the cost can be cyclically decreased.

We also study when convergence of compactly supported measures in L^p transportation distance implies convergence in L^∞ transportation distance. For measures with connected supports this property is characterized by uniform lower bounds on the measures of balls centered at the supports of the measures or, equivalently, by the Hausdorff-convergence of the supports.

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1. Introduction

Suppose we are given two Borel probability measures $\mu, \nu \in \mathscr{P}(X)$ on a metric space (X, d) and a function $c\colon X \times X \to [-\infty, \infty]$ representing the cost of moving mass. The optimal mass transportation problem, in the Kantorovich formulation, is then to minimize the quantity

$$\int_{X \times X} c(x, y)\, \mathrm{d}\lambda(x, y) \tag{1.1}$$

over all possible transport plans $\lambda \in \Pi(\mu, \nu)$, i.e. Borel probability measures in $X \times X$ having the marginals μ and ν. An optimal transport plan λ minimizing (1.1) exists under mild regularity assumptions, for example if the cost function c is lower semicontinuous and bounded from below and if the metric space (X, d) is complete and separable [13, Theorem 4.1]. Under much more restrictive assumptions such minimizer is unique and given by an optimal transport map $T\colon X \to X$ as $\lambda = (\mathrm{id}, T)_\sharp \mu$.

Often the cost function in (1.1) is of the form $c(x, y) = h(\mathsf{d}(x, y))$ with some convex function $h\colon [0, \infty) \to [0, \infty)$. The most commonly used cost functions are the p-th powers of the distance with $p \in [1, \infty)$. This leads to the L^p transportation distances W_p defined between $\mu, \nu \in \mathscr{P}(X)$ by

$$W_p(\mu, \nu) = \inf_{\lambda \in \Pi(\mu, \nu)} \left(\int \mathsf{d}^p(x, y)\, \mathrm{d}\lambda(x, y) \right)^{1/p}.$$

It is well known that the W_p distance metrizes the topology of weak convergence (up to convergence of p-th moments). The W_p distances with $p \in (1, \infty)$ are often easier to handle, for instance due to strict convexity, than the limiting cases $p = 1$ and $p = \infty$. In the latter one the distance is defined as

$$W_\infty(\mu, \nu) = \inf_{\lambda \in \Pi(\mu, \nu)} \lambda - \operatorname*{ess\,sup}_{(x,y) \in X^2} \mathsf{d}(x, y).$$

The distance W_∞ is even more cumbersome than W_1. This is because the problem of infimizing the cost

$$\lambda - \operatorname*{ess\,sup}_{(x,y) \in X^2} \mathsf{d}(x, y)$$

over all $\lambda \in \Pi(\mu, \nu)$ is not convex and thus it is not additive. Consequently, restrictions of optimal transport plans for W_∞ are not necessarily optimal. The problem of restrictions in W_∞ was addressed by Champion, De Pascale and Juutinen in [6] where they introduced the notion of *restrictable solutions*. Those are the optimal transport plans that retain optimality under restrictions. Restrictable solutions appear as the limit solutions in the approximation as $p \to \infty$ and, more generally, can be characterized by a