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Entropy and approximation numbers of weighted Sobolev spaces via bracketing ☆

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ABSTRACT

We investigate the asymptotic behaviour of entropy and approximation numbers of the compact embedding id : $E^m_{p,\sigma}(B) \hookrightarrow L_p(B)$, $1 \leq p < \infty$, defined on the unit ball B in $\mathbb{R}^n$. Here $E^m_{p,\sigma}(B)$ denotes a Sobolev space with a power weight perturbed by a logarithmic function. The weight contains a singularity at the origin. Inspired by Evans and Harris [5], we apply a bracketing technique which is an analogue to that of Dirichlet–Neumann bracketing used by Triebel in [14] for $p = 2$.

1. Introduction

This paper deals with the compact embedding

$$\mathrm{id} : E^m_{p,\sigma}(B) \hookrightarrow L_p(B), \qquad 1 \leq p < \infty,\ m \in \mathbb{N},\ \sigma > 0, \tag{1.1}$$

where $E^m_{p,\sigma}(B)$ is the closure of $C^m_0(B)$ with respect to the norm

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$$\|f|E^m_{p,\sigma}(B)\| := \left(\int_B |x|^{mp}(1+|\log|x||)^{\sigma p} \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \right)^{1/p} \tag{1.2}$$

and $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit ball. Problems of this type have been considered so far in [14,8,9]. In case of Hilbert spaces Triebel obtained in [14] sharp results for the corresponding entropy and approximation numbers. Namely, if $p = 2$ it holds for $k \in \mathbb{N}$, $k \geq 2$, that

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}} & \text{if } \sigma > \frac{m}{n} \\ k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}} & \text{if } \sigma = \frac{m}{n} \\ k^{-\sigma} & \text{if } 0 < \sigma < \frac{m}{n}. \end{cases} \tag{1.3}$$

We will extend this result and confirm Triebel's Conjecture 3.8 in [14] that (1.3) holds for all $1 \leq p < \infty$. In [14] the so-called Courant–Weyl method of Dirichlet–Neumann bracketing is used. This technique is not available for $p \neq 2$, but a partial analogue was established by Evans and Harris in [5]. These authors deal with Sobolev spaces $W^1_p(\Omega)$ on a wide class of domains, i.e. rooms and passages domains or generalised ridged domains. We want to transfer this idea to control the singularity in our situation.

The paper is organised as follows. In Section 2 we collect basic notation and briefly introduce the setting of the compact embedding (1.1). In Section 3 we present a bracketing method to determine the asymptotic behaviour of the number

$$\nu_0(\varepsilon, \Omega) := \max\{k \in \mathbb{N} : a_k(\mathrm{id}_\Omega) \geq \varepsilon\}, \qquad \varepsilon > 0,$$

as $\varepsilon \to 0$. The operator id_Ω denotes the restriction of id to a subset $\Omega \subset B$. Let $\Omega = \left(\bigcup_{j=1}^J \overline{\Omega}_j\right)^\circ$ with disjoint domains Ω_j. The essence of the method is the bracketing property stated in Proposition 3.3, which reads as

$$\sum_{j=1}^J \mu_0(\varepsilon, \Omega_j) \leq \nu_0(\varepsilon, \Omega) \leq \sum_{j=1}^J \nu_0(\varepsilon, \Omega_j). \tag{1.4}$$

Here the number $\mu_0(\varepsilon, \Omega)$, $\Omega \subseteq B$, is defined by

$$\mu_0(\varepsilon, \Omega) := \max\left\{ \dim S : \alpha(S) = \sup_{u \in S\setminus\{0\}} \frac{\|u|E^m_{p,\sigma}(\Omega)\|}{\|u|L_p(\Omega)\|} \leq \frac{1}{\varepsilon} \right\}$$

where the maximum is taken over all finite-dimensional linear subspaces S of $E^m_{p,\sigma}(\Omega)$. This approach attempts to mimic the Dirichlet–Neumann bracketing described in [8,9,14] if $p = 2$. This is emphasised by Proposition 3.4. We use (1.4) to cut off the singularity of the weight $b_{m,\sigma}(x) = |x|^{mp}(1 + |\log|x||)^{\sigma p}$, $x \in B$, at the origin and consider the corresponding domains separately. In this way we achieve in Proposition 3.6