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# Diagonalization of bosonic quadratic Hamiltonians by Bogoliubov transformations

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ABSTRACT

We provide general conditions for which bosonic quadratic Hamiltonians on Fock spaces can be diagonalized by Bogoliubov transformations. Our results cover the case when quantum systems have infinite degrees of freedom and the associated one-body kinetic and paring operators are unbounded. Our sufficient conditions are optimal in the sense that they become necessary when the relevant one-body operators commute.



## 1. Introduction

We consider Hamiltonians on Fock space which are quadratic in terms of bosonic creation and annihilation operators. In many cases the quadratic Hamiltonians can be diagonalized by Bogoliubov transformations, namely they can be transformed to those of noninteracting particles by a special class of unitary operators which preserve the

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CCR algebra. The aim of our present work is to give rigorous conditions for which the diagonalization can be carried out for quantum systems of infinite degrees of freedom where the kinetic and paring operators are unbounded.

### 1.1. Quadratic Hamiltonian

Let us introduce the mathematical setting. Our one-body Hilbert space $\mathfrak{h}$ is a complex separable Hilbert space with inner product $\langle .,. \rangle$ which is linear in the second variable and anti-linear in the first. In the grand canonical ensemble the number of particles is not fixed and it is natural to introduce the (bosonic) Fock space

$$\mathcal{F}(\mathfrak{h}) := \bigoplus_{N=0}^{\infty} \bigotimes_{\text{sym}}^{N} \mathfrak{h} = \mathbb{C} \oplus \mathfrak{h} \oplus (\mathfrak{h} \otimes_s \mathfrak{h}) \oplus \cdots .$$

Noninteracting systems are described by the Hamiltonians of the form

$$\mathrm{d}\Gamma(h) := \bigoplus_{N=0}^{\infty} (\sum_{j=1}^{N} h_j) = 0 \oplus h \oplus (h \otimes 1 + 1 \otimes h) \oplus \cdots$$

on the Fock space, where $h > 0$ is a self-adjoint operator on $\mathfrak{h}$. Although we will work in an abstract setting, the reader may keep in mind the typical example that $h = -\Delta + V(x)$ on $\mathfrak{h} = L^2(\mathbb{R}^d)$, where $V$ is an external potential which serves to bind the particles. The operator $\mathrm{d}\Gamma(h)$ is well-defined on the core

$$\bigcup_{M \geq 0} \bigoplus_{n=0}^{M} \bigotimes_{\text{sym}}^{n} D(h)$$

and it can be extended to a positive self-adjoint operator on Fock space by Friedrichs' extension. The spectrum of $\mathrm{d}\Gamma(h)$ is nothing but the closure of the finite sums of elements of the spectrum of $h$. In particular, the spectrum of the particle number operator $\mathcal{N} := \mathrm{d}\Gamma(1)$ is $\{0, 1, 2, \ldots\}$.

In many physical situations the interaction between particles plays a crucial role and it complicates the picture dramatically. In principle, solving interacting systems exactly is mostly unrealistic and certain approximations are necessary. In the celebrated 1947 paper [4], Bogoliubov introduced an approximation theory for a weakly interacting Bose gas where the many-body system is effectively described by a *quadratic Hamiltonian* on Fock space, which will be described below. We refer to the book [16] for a pedagogical introduction to Bogoliubov's approximation. Bogoliubov's theory has been justified rigorously in various situations including the ground state energy of one and two-component Bose gases [17,18,25], the Lee–Huang–Yang formula of dilute gases [8,10,27] and the excitation spectrum in the mean-field limit [22,11,15,7,20].