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Fourier coefficients of automorphic forms and integrable discrete series

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ABSTRACT

Let G be the group of $\mathbb{R}$-points of a semisimple algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$. Assume that G is connected and noncompact. We study Fourier coefficients of Poincaré series attached to matrix coefficients of integrable discrete series. We use these results to construct explicit automorphic cuspidal realizations, which have appropriate Fourier coefficients $\neq 0$, of integrable discrete series in families of congruence subgroups. In the case of $G = Sp_{2n}(\mathbb{R})$, we relate our work to that of Li [14]. For $\mathcal{G}$ quasi-split over $\mathbb{Q}$, we relate our work to the result about Poincaré series due to Khare, Larsen, and Savin [12].

1. Introduction

Let G be the group of $\mathbb{R}$-points of a semisimple algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$. Assume that G is connected and noncompact. Let K be its maximal compact subgroup, $\mathfrak{g}$ be the real Lie algebra of G, and $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ the center of the universal enveloping algebra of the complexification of $\mathfrak{g}$. In this paper we assume that $rank(K) = rank(G)$ so

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that G admits discrete series. Let Γ be a discrete subgroup of finite covolume in G. The problem of (mostly asymptotic) realization of discrete series as subrepresentations in the discrete part of the spectrum of $L^2(\Gamma\backslash G)$ has been studied extensively using various methods such as cohomology of discrete subgroups and adelic Arthur trace formula [23,6,22,7,8]. But Fourier coefficients of such realizations are not well understood. On the other hand, assuming that $\mathcal{G}$ is a quasi-split almost simple algebraic group over $\mathbb{Q}$, for any appropriate generic integrable discrete series π of G, Khare, Larsen, and Savin ([12] Theorem 4.5) construct globally generic automorphic cuspidal representation W of $\mathcal{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$, such that its Archimedean component is π i.e., $W_\infty = \pi$. They use classical theory of Poincaré series attached to matrix coefficients of π which are $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$-finite and K-finite on the right extended to adelic settings [1,3,2]. A detailed treatment of such series and criteria for being $\neq 0$ in the adelic settings can be found in [18]. In ([12] Theorem 4.5), Khare, Larsen, and Savin in fact prove an analogue of a well-known result of Vignéras, Henniart and Shahidi [24] for generic supercuspidal representations of semisimple p-adic groups. ([12] Theorem 4.5) is used to study problems in inverse Galois theory. On the other hand, (possibly degenerate) Fourier coefficients of automorphic forms are important for the theory of automorphic L-functions [24,9,25]. So, it is important to study (possibly degenerate) Fourier coefficients of Poincaré series attached to matrix coefficients of integrable discrete series which are K-finite. This is the goal of the present paper. The techniques used in this paper are refinements of those of Khare, Larsen, and Savin used in the proof of ([12] Theorem 4.5), and those of (for adelic or for real groups) used in [19,17,20,21]. We not only improve the results in generic case (see Theorems 6-7 and 6-9) but we also construct very explicit degenerate cuspidal automorphic models of integrable discrete series [14] for $Sp_{2n}(\mathbb{R})$ (see Theorem 6-12).

The paper actually has two parts: preliminary local Archimedean part (Sections 2, 3, Appendix A), and main cuspidal automorphic part (Sections 4, 5, 6). Integrable discrete series for G are analogues of supercuspidal representations for semi-simple p-adic groups. In Section 3 we refine and generalize ([12], Proposition 4.2) discussing the analogue of the following simple result for p-adic groups which we explain in detail. Assume for the moment that G is a semisimple p-adic group, U closed unimodular subgroup of G, and $\chi : U \longrightarrow \mathbb{C}^\times$ a unitary character. Let $(\pi, \mathcal{H})$ be an irreducible supercuspidal representation acting on the Hilbert space $\mathcal{H}$. Let us write $\mathcal{H}^\infty$ for the space of smooth vectors (i.e., the subspace of all vectors in $\mathcal{H}$ which have open stabilizers). The space $\mathcal{H}^\infty$ is usual smooth irreducible algebraic representation of G. We say that π is (χ, U)-generic if there exists a non-zero (algebraic) functional $\lambda : \mathcal{H}^\infty \longrightarrow \mathbb{C}$ which satisfies $\lambda(\pi(u)h) = \chi(u)\lambda(h)$, $u \in U$, $h \in \mathcal{H}^\infty$. Let φ be a non-zero matrix coefficient of $(\pi, \mathcal{H}^\infty)$. Then, $\varphi \in C_c^\infty(G)$. Next, a simple argument (see Lemma 3-7), first observed by Miličić in his unpublished lecture notes about $SL_2(\mathbb{R})$, Schur orthogonality relations show that for each $h \in \mathcal{H}^\infty$, we can select a matrix coefficient φ such that $\pi(\overline{\varphi})h = h$. Now, since $h = \pi(\overline{\varphi})h = \int_G \overline{\varphi(g)}\pi(g)hdg$ is essentially a finite sum, we have