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Hecke–Bochner identity and eigenfunctions associated to Gelfand pairs on the Heisenberg group ☆

Amit Samanta

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

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A B S T R A C T

Let $\mathbb{H}^n$ be the $(2n+1)$-dimensional Heisenberg group, and let K be a compact subgroup of $U(n)$, such that $(K, \mathbb{H}^n)$ is a Gelfand pair. Also assume that the K-action on $\mathbb{C}^n$ is polar. We prove a Hecke–Bochner identity associated to the Gelfand pair $(K, \mathbb{H}^n)$. For the special case $K = U(n)$, this was proved by Geller [6], giving a formula for the Weyl transform of a function f of the type $f = Pg$, where g is a radial function, and P a bigraded solid $U(n)$-harmonic polynomial. Using our general Hecke–Bochner identity we also characterize (under some conditions) joint eigenfunctions of all differential operators on $\mathbb{H}^n$ that are invariant under the action of K and the left action of $\mathbb{H}^n$.

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E-mail address: samanta.amit@sns.it.

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1. Introduction

This paper is concerned with two fundamental problems in Harmonic analysis on the Heisenberg group, $\mathbb{H}^n$. The first one is the Hecke–Bochner identity and the second one is a characterization of joint eigenfunctions for a certain family of invariant differential operators on $\mathbb{H}^n$. We first briefly recall the known results in this direction.

The Hecke–Bochner identity on $\mathbb{R}^n$ states that (see [16, Theorem 3.10, p. 158]) the Fourier transform of a function $f = Pg$, where P is a homogeneous solid $SO(n)$-harmonic polynomial (of degree k say) and g is radial, is given by $\widehat{Pg} = Ph$, where h is a radial function given by

$$h(r) = i^{-k} \int\limits_{s=0}^{\infty} g(s) \frac{J_{\frac{n}{2}+k-1}(rs)}{(rs)^{\frac{n}{2}+k-1}} s^{n+k-1} ds,$$

where $J_{\frac{n}{2}+k-1}$ is the Bessel function of order $\frac{n}{2}+k-1$. Secondly, any eigenfunction φ of $\triangle$, the Laplacian on $\mathbb{R}^n$, with eigenvalue $-\lambda^2$ is given by the integral representation

$$\varphi(x) = \int\limits_{S^{n-1}} e^{i\lambda x\cdot\omega} dT(\omega),$$

where T is a certain analytic functional. See Helgason [10, Theorem 2.1, p. 5] for $n = 2$ and Hashizume et al. [7] for general case. Both these results can be interpreted in terms of harmonic analysis on the Gelfand pair $\big(\mathbb{R}^n \ltimes SO(n), SO(n)\big)$. Note that a solid homogeneous harmonic polynomial of degree k is an element which transforms according to a class one representation of $SO(n)$. Next, the Laplacian $\triangle$ is the generator of $\mathbb{R}^n \ltimes SO(n)$ invariant differential operators on $\mathbb{R}^n$. This point of view has a natural generalization to other homogeneous spaces.

In the context of Riemannian symmetric spaces $X = G/K$, Helgason [9, Corollary 7.4] characterized all K-finite joint eigenfunctions for $D(G/K)$. The characterization of arbitrary joint eigenfunctions for $D(G/K)$ was done by Helgason [8, Chapter IV, Corollary 1.6] when rank $X = 1$ and by Kashiwara et al. [12] in the general case. A Hecke–Bochner type identity was established, when X is of rank one, by Bray [3]. For general case, see [11, Chapter III, Corollary 5.5].

In this paper, we consider these two questions on the Heisenberg group associated to the Gelfand pair $(K, \mathbb{H}^n)$, where $K \subset U(n)$ and the K-action on $\mathbb{C}^n$ is polar. We prove a Hecke–Bochner type identity (Theorem 7.4), giving a formula for the Weyl transform of a function which transforms according to a class one representation of K. We will see that the formula involves generalized K-spherical functions, as in the case of Euclidean spaces and Riemannian symmetric spaces. For the special case $K = U(n)$ this was already proved by Geller [6, Theorem 4.2]. Let $\mathcal{L}_K(\mathbf{h}_n)$ be the algebra of all differential operators on $\mathbb{H}^n$ that are invariant under the action of K and the left action of $\mathbb{H}^n$. Any joint eigenfunction of all $D \in \mathcal{L}_K(\mathbf{h}_n)$ has to be of the form $f(z,t) = e^{i\lambda t} g(z)$