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Non-hyperbolic closed geodesics on positively curved Finsler spheres

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ABSTRACT

In this paper, we prove that for every Finsler n-dimensional sphere $(S^n, F), n \geq 3$ with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$, there exist at least three distinct closed geodesics and at least two of them are elliptic if the number of prime closed geodesics is finite. When $n \geq 6$, these three distinct closed geodesics are non-hyperbolic.

1. Introduction and main result

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime* if it is not a multiple covering

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(i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. Note that unlike Riemannian manifold, the inverse curve c^{-1} of a closed geodesic c on an irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0,1)$ such that $c(t) = d(t+\theta)$ for all $t \in \mathbf{R}$. On a reversible Finsler (or Riemannian) manifold, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., their image sets in M are distinct. We shall omit the word *distinct* when we talk about more than one prime closed geodesic.

For a closed geodesic c on n-dimensional manifold (M, F), denote by P_c the linearized Poincaré map of c. Then $P_c \in \mathrm{Sp}(2n-2)$ is symplectic. For any $M \in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C} |\ |z| = 1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2k$. A closed geodesic c is called *elliptic* if $e(P_c) = 2(n-1)$, i.e., all the eigenvalues of P_c locate on $\mathbf{U}$; *hyperbolic* if $e(P_c) = 0$, i.e., all the eigenvalues of P_c locate away from $\mathbf{U}$; *non-degenerate* if 1 is not an eigenvalue of P_c. A Finsler manifold (M, F) is called *bumpy* if all the closed geodesics on it are non-degenerate.

There is a famous conjecture in Riemannian geometry which claims the existence of infinitely many closed geodesics on any compact Riemannian manifold. This conjecture has been proved for many cases, but not yet for compact rank one symmetric spaces except for S^2. The results of Franks [13] in 1992 and Bangert [3] in 1993 imply that this conjecture is true for any Riemannian 2-sphere (cf. [15] and [16]). But once one moves to the Finsler case, the conjecture becomes false. It was quite surprising when Katok [17] in 1973 found some irreversible Finsler metrics on spheres with only finitely many closed geodesics and all closed geodesics are non-degenerate and elliptic (cf. [34]).

Recently, index iteration theory of closed geodesics (cf. [5] and [21]) has been applied to study the closed geodesic problem on Finsler manifolds. For example, Bangert and Long in [4] show that there exist at least two closed geodesics on every (S^2, F). After that, a great number of multiplicity and stability results have appeared (cf. [8–12,22,23,28–33] and the references therein).

In [27], Rademacher has introduced the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold defined by

$$\lambda = \max\{F(-X) \mid X \in TM,\ F(X) = 1\} \geq 1.$$

Then Rademacher in [28] has obtained some results about multiplicity and the length of closed geodesics and about their stability properties. For example, let F be a Finsler metric on S^n with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1$, then there exist at least $n/2 - 1$ closed geodesics with length $< 2n\pi$. If $\frac{9\lambda^2}{4(1+\lambda)^2} < K \leq 1$ and $\lambda < 2$, then there exists a closed geodesic of elliptic-parabolic, i.e., its linearized Poincaré map split into 2-dimensional rotations and a part whose eigenvalues are ± 1. Some similar results in the Riemannian case are obtained in [1] and [2].