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Singular integral operators with kernels associated to negative powers of real-analytic functions ☆

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ABSTRACT

Given a real-analytic function $b(x)$ defined on a neighborhood of the origin with $b(0) = 0$, we consider local convolutions with kernels which are bounded by $|b(x)|^{-a}$, where $a > 0$ is the smallest number for which $|b(x)|^{-a}$ is not integrable on any neighborhood of the origin. Under appropriate first derivative bounds and a cancellation condition, we prove L^p boundedness theorems for such operators including when the kernel is not integrable. We primarily (but not exclusively) consider the $p = 2$ situation. The operators considered generalize both local versions of Riesz transforms and some local multiparameter singular integrals. Generalizations of our results to nontranslation-invariant versions as well as singular Radon transform versions are also proven.

1. Introduction and theorems in the multiplicity one case

Let $n \geq 2$ and let $b(x)$ be a real-analytic function on a neighborhood of the origin in $\mathbf{R}^n$ with $b(0) = 0$. By resolution of singularities, there is a number $\delta_0 > 0$ such

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that on any sufficiently small neighborhood U of the origin, $\int_U |f|^{-\delta} = \infty$ for $\delta \geq \delta_0$, and $\int_U |f|^{-\delta} < \infty$ for $\delta < \delta_0$. The number δ_0 is sometimes referred to as the "critical integrability exponent" of f at the origin. In this paper, we consider operators of the form

$$Tf(x) = \int_{\mathbf{R}^n} f(x-y)\,\alpha(x,y)\,m(y)\,|b(y)|^{-\delta_0}\,dy \tag{1.1}$$

Here $\alpha(x,y)$ is a Schwartz function, and $m(y)$ is a bounded real-valued function on a neighborhood of the origin such that $m(y)|b(y)|^{-\delta_0}$ satisfies natural derivative and cancellation conditions deriving from $b(y)$ that allows T to be considered as a type of singular integral operator. The focus of this paper will be to determine the boundedness properties of such T on L^p spaces for $1 < p < \infty$. Most of our results will concern the L^2 situation. As we will see, the operators we will consider will generalize local singular integral operators such as local versions of Riesz transforms, and also classes of local multiparameter singular integrals.

We will see that some of our proofs immediately extend to analogues of singular Radon transforms for such singular integral operators. Namely, our results will cover some operators of the following form, where $x \in \mathbf{R}^m$ and h is a real-analytic map from a neighborhood of the origin in $\mathbf{R}^n$ into $\mathbf{R}^m$ with $h(0) = 0$.

$$T'f(x) = \int_{\mathbf{R}^n} f(x-h(y))\,\alpha(x,y)\,m(y)\,|b(y)|^{-\delta_0}\,dy \tag{1.2}$$

To help define what types of kernels we allow, we now delve into the resolution of singularities near the origin of a real-analytic function $b(x)$ with $b(0) = 0$. For this we use the resolution of singularities theorem of [4], but other resolution of singularities theorems including Hironaka's famous work [7,8] can be used in similar ways.

By [4], there is a neighborhood U of the origin such that there exist finitely many coordinate change maps $\{\beta_i(x)\}_{i=1}^M$ and finitely many vectors $\{(m_{i1}, \ldots, m_{in})\}_{i=1}^M$ of nonnegative integers such that if $\rho(x)$ is a nonnegative smooth bump function supported in U with $\rho(0) \neq 0$, then $\rho(x)$ can be written in the form $\rho(x) = \sum_{i=1}^M \rho_i(x)$ in such a way that each $\rho_i \circ \beta_i(x)$, after an adjustment on a set of measure zero, is a smooth nonnegative bump function on a neighborhood of the origin with $\rho_i \circ \beta_i(0) \neq 0$. The components of each $\beta_i(x)$ are real-analytic. In addition, β_i is a bijection from $\{x : \rho_i \circ \beta_i(x) \neq 0, x_i \neq 0$ for all $i\}$ to $\{x : \rho_i(x) \neq 0\} - Z_i$ where Z_i has measure zero, and on a connected neighborhood U_i of the support of $\rho_i \circ \beta_i(x)$ the function $b \circ \beta_i(x)$ is well-defined and "comparable" to the monomial $x_1^{m_{i1}} \ldots x_n^{m_{in}}$, meaning that there is a real-analytic function $c_i(x)$ with $|c_i(x)| > \epsilon > 0$ on U_i such that $b \circ \beta_i(x) = c_i(x) x_1^{m_{i1}} \ldots x_n^{m_{in}}$ on U_i. This decomposition is such that the Jacobian determinant of $\beta_i(x)$ can be written in an analogous form $d_i(x) x_1^{e_{i1}} \ldots x_n^{e_{in}}$ on U_i; again the e_{ij} are integers and $|d_i(x)| > \epsilon > 0$ on U_i.

In view of the above, one has