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Density and duality theorems for regular Gabor frames

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ABSTRACT

We investigate Gabor frames on locally compact abelian groups with time–frequency shifts along non-separable, closed subgroups of the phase space. Density theorems in Gabor analysis state necessary conditions for a Gabor system to be a frame or a Riesz basis, formulated only in terms of the index subgroup. In the classical results the subgroup is assumed to be discrete. We prove density theorems for general closed subgroups of the phase space, where the necessary conditions are given in terms of the "size" of the subgroup. From these density results we are able to extend the classical Wexler–Raz biorthogonal relations and the duality principle in Gabor analysis to Gabor systems with time–frequency shifts along non-separable, closed subgroups of the phase space. Even in the euclidean setting, our results are new.

1. Introduction

Classical harmonic analysis on locally compact abelian (LCA) groups provides a natural framework for many of the topics considered in modern time–frequency analysis. The

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setup is as follows. Let $(G, \cdot)$ denote a second countable LCA group, and let $(\widehat{G}, \cdot)$ denote its dual group, consisting of all characters. One then defines the translation operator T_λ, $\lambda \in G$, as

$$T_\lambda : L^2(G) \to L^2(G),\ (T_\lambda f)(x) = f(x\lambda^{-1}), \quad x \in G,$$

and the modulation operator E_γ, $\gamma \in \widehat{G}$, as

$$E_\gamma : L^2(G) \to L^2(G),\ (E_\gamma f)(x) = \gamma(x) f(x), \quad x \in G.$$

The central objects of this work are so-called regular Gabor systems in $L^2(G)$ with modulation and translation along a closed subgroup Δ of $G \times \widehat{G}$ generated by a window function $g \in L^2(G)$; this is a collection of functions of the following form:

$$\mathscr{G}(g, \Delta) := \{\pi(\nu)g\}_{\nu \in \Delta}, \quad \text{where } \pi(\nu) := E_\gamma T_\lambda \text{ for } \nu = (\lambda, \gamma) \in G \times \widehat{G}.$$

The tensor product $G \times \widehat{G}$ is called the phase–space or the time–frequency plane, and $\pi(\nu)g$ is a time–frequency shift of g.

We are interested in linear operators of the form

$$C_{g,\Delta} : L^2(G) \to L^2(\Delta), \quad C_{g,\Delta} f = \nu \mapsto \langle f, \pi(\nu)g \rangle$$

as well as their left-inverses (if they exist) and their adjoint operators. The $C_{g,\Delta}$ transform is called an *analysis* operator, while its adjoint is called *synthesis*. In the analysis process $C_{g,\Delta} f$ of a function $f \in L^2(G)$, we obtain information of the time–frequency content in the function f.

If the operator $C_{g,\Delta}$ is bounded below and above, we say that $\mathscr{G}(g, \Delta)$ is a Gabor frame for $L^2(G)$. In case the two constants from these bounds can be taken to be equal, we say that $\mathscr{G}(g, \Delta)$ is a tight frame; if they can be taken to be equal to one, $\mathscr{G}(g, \Delta)$ is said to be a Parseval frame. One can show that the property of being a frame allows for stable reconstruction of any $f \in L^2(G)$ from its time–frequency information given by $C_{g,\Delta} f$. In particular, if $C_{g,\Delta}$ is bounded from below and above, then there exists another function $h \in L^2(G)$ such that $C_{h,\Delta}$ is a bounded operator and such that

$$\langle f_1, f_2 \rangle = \int_\Delta C_{g,\Delta} f_1(\nu)\, \overline{C_{h,\Delta} f_2(\nu)}\, d\nu$$

for all $f_1, f_2 \in L^2(G)$, where $d\nu$ denotes the Haar measure on Δ. Two such Gabor systems $\mathscr{G}(g, \Delta)$ and $\mathscr{G}(h, \Delta)$ are said to be *dual* Gabor frames. If $\mathscr{G}(g, \Delta)$ is a frame with $C_{g,\Delta}$ being surjective, we say that $\mathscr{G}(g, \Delta)$ is a *Riesz family*.

In case $\Delta = G \times \widehat{G}$ the analysis operator $C_{g,\Delta}$ is the well-known *short-time Fourier transform*, usually written $\mathcal{V}_g$, which is an isometry for any window function $g \in L^2(G)$ satisfying $\|g\| = 1$. In the language of frame theory, $\mathscr{G}(g, G \times \widehat{G})$ is said to be a Parseval