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Noncommutative uncertainty principles

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ABSTRACT

The classical uncertainty principles deal with functions on abelian groups. In this paper, we discuss the uncertainty principles for finite index subfactors which include the cases for finite groups and finite dimensional Kac algebras. We prove the Hausdorff–Young inequality, Young's inequality, the Hirschman–Beckner uncertainty principle, the Donoho–Stark uncertainty principle. We characterize the minimizers of the uncertainty principles and then we prove Hardy's uncertainty principle by using minimizers. We also prove that the minimizer is uniquely determined by the supports of itself and its Fourier transform. The proofs take the advantage of the analytic and the categorial perspectives of subfactor planar algebras. Our method to prove the uncertainty principles also works for more general cases, such as Popa's λ-lattices, modular tensor categories, etc.

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1. Introduction

In quantum mechanics, Heisenberg's uncertainty principle says that the more precisely the position of some particle is determined, the less precisely its momentum can be known. This uncertainty principle is a fundamental result in quantum mechanics. By using the Shannon entropy of the measurement, entropic uncertainty principles were established, which strengthen and generalize Heisenberg's uncertainty principle. The phenomenon of "uncertainty" also appears in mathematics. In Harmonic Analysis, various uncertainty principles are found by mathematicians such as Hardy's uncertainty principle [14], Hirschman's uncertainty principle [15] etc. In information theory, Donoho and Stark [11] introduced an uncertainty principle. They proved that a signal may be reconstructed with a few samples. This idea was further developed by Candes, Romberg and Tao [9] in the theory of compressed sensing.

The classical uncertainty principles deal with functions on an abelian group and its dual. A natural question is to ask whether there exist uncertainty principles for a non-abelian group. Recently the uncertainty principle of finite groups [13], compact groups [1] and Kac algebras (C^* Hopf algebras or quantum groups in von Neumann algebraic setting) [8] were discussed from different perspectives. It is natural to consider the uncertainty principles of Kac algebras due to the existence of a proper pair of observables. There is a one to one correspondence between Kac algebras and a special family of subfactors, namely irreducible *depth-2* subfactors [29]. Subfactors naturally provide vector spaces with two or more observables. In this paper, we are going to talk about the uncertainty principles of subfactors.

Modern subfactor theory was initiated by Jones. Subfactors generalize the symmetries of groups and their duals. The index of a subfactor generalizes the order of a group. All possible indices of subfactors,

$$\{4\cos^2\frac{\pi}{n}, n=3,4,\cdots\}\cup[4,\infty],$$

were found by Jones in his remarkable rigidity result [18]. A deep theorem of Popa [26] showed that the *standard invariant*, a $\mathbb{Z}_2$ graded filtered algebra, is a complete invariant of *finite depth* subfactors of the hyperfinite factor of type II_1. There are three axiomatizations of the standard invariant: Ocneanu's paragroups [24]; Popa's standard λ-lattices [27]; Jones' subfactor planar algebras [17].

In terms of paragroups, the n-box space of the standard invariant of a (finite index, finite depth, irreducible) subfactor consists of linear sums of certain length-$2n$ loops of a bipartite graph, the *principal graph* of the subfactor. The Fourier transform of paragroups was introduced by Ocneanu [24] as the 1-click rotation of loops. It generalizes the Fourier transform of finite groups. Note that the Fourier transform of the n-box space has periodicity $2n$. Up to an anti-isomorphism, the n-box space admits n^*-algebraic structures and a proper n-tuple of observables. We will prove the uncertainty principles of n-box spaces. In particular, if we start with a subfactor arising from an outer action