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Entropy dissipation estimates for the Landau equation in the Coulomb case and applications

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A B S T R A C T

We present in this paper an estimate which bounds from below the entropy dissipation $D(f)$ of the Landau operator with Coulomb interaction by a weighted H^1 norm of the square root of f. As a consequence, we get a weighted $L^1_t(L^3_v)$ estimate for the solutions of the spatially homogeneous Landau equation with Coulomb interaction, and the propagation of L^1 moments of any order for this equation. We also present an application of our estimate to the Landau equation with (moderately) soft potentials, providing thus a new proof of some recent results of [30].

1. Introduction and main result

1.1. Description of the Landau equation

We recall the spatially homogeneous Landau equation of plasma theory (cf. [8,22]),

$$\frac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v), \qquad v \in \mathbb{R}^N, \quad t \geq 0, \tag{1}$$

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where $f := f(t,v)$ is a nonnegative function representing the number of particles which at time t move with velocity v, and Q is a nonlinear quadratic operator modeling the collisions between those particles. It acts on the variable v only, and writes

$$Q(f,f)(v) = \sum_{i=1}^{N} \frac{\partial}{\partial v_i} \left\{ \int_{\mathbb{R}^N} \sum_{j=1}^{N} a_{ij}(v-w) \left(f(w)\, \frac{\partial f}{\partial v_j}(v) - f(v)\, \frac{\partial f}{\partial v_j}(w) \right) dw \right\}. \tag{2}$$

We also introduce the (nonnegative) initial datum $f_{in} : \mathbb{R}^N \to \mathbb{R}_+$:

$$\forall v \in \mathbb{R}^N, \qquad f(0,v) = f_{in}(v). \tag{3}$$

Here, $(a_{ij}(z))_{ij}$ ($z \in \mathbb{R}^N$) is given by

$$a_{ij}(z) = \Pi_{ij}(z)\, \psi(|z|), \tag{4}$$

where ψ is a nonnegative function, and

$$\Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2} \tag{5}$$

is the i,j-component of the orthogonal projection Π onto $z^\perp := \{y / y \cdot z = 0\}$.

It is customary to define the following functions:

$$b_i(z) = \sum_{j=1}^{N} \partial_j a_{ij}(z) = -(N-1)\, \frac{z_i}{|z|^2}\, \psi(|z|), \tag{6}$$

$$c(z) = \sum_{i=1}^{N} \sum_{j=1}^{N} \partial_{ij} a_{ij}(z) = -(N-1) \left((N-2)\, \frac{\psi(|z|)}{|z|^2} + \frac{\psi'(|z|)}{|z|} \right). \tag{7}$$

The computation above must be adapted when ψ has strong singularities (for example, in the important case when $N = 3$ and $\psi(|z|) = |z|^{-1}$, corresponding to the Coulomb interaction, one finds $c(z) = -8\pi\, \delta_0$).

Using those functions, the Landau operator can be written (at the formal level) as a (conservative or non-conservative) nonlinear (quadratic) parabolic equation with nonlocal coefficients, indeed

$$Q(f,f) = \sum_{i=1}^{N} \frac{\partial}{\partial v_i} \left(\sum_{j=1}^{N} (a_{ij} * f)\, \frac{\partial f}{\partial v_j} - (b_i * f)\, f \right), \tag{8}$$

and