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Gabor orthonormal bases generated by the unit cubes

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ABSTRACT

We consider the problem in determining the countable sets Λ in the time-frequency plane such that the Gabor system generated by the time-frequency shifts of the window $\chi_{[0,1]^d}$ associated with Λ forms a Gabor orthonormal basis for $L^2(\mathbb{R}^d)$. We show that, if this is the case, the translates by elements Λ of the unit cube in $\mathbb{R}^{2d}$ must tile the time-frequency space $\mathbb{R}^{2d}$. By studying the possible structure of such tiling sets, we completely classify all such admissible sets Λ of time-frequency shifts when $d = 1, 2$. Moreover, an inductive procedure for constructing such sets Λ in dimension $d \geq 3$ is also given. An interesting and surprising consequence of our results is the existence, for $d \geq 2$, of discrete sets Λ with $\mathcal{G}(\chi_{[0,1]^d}, \Lambda)$ forming a Gabor orthonormal basis but with the associated "time"-translates of the window $\chi_{[0,1]^d}$ having significant overlaps.

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1. Introduction

Let g be a non-zero function in $L^2(\mathbb{R}^d)$ and let Λ be a discrete countable set on $\mathbb{R}^{2d}$, where we identify $\mathbb{R}^{2d}$ to the time-frequency plane by writing $(t, \lambda) \in \Lambda$ with $t, \lambda \in \mathbb{R}^d$. The Gabor system associated with the window g consists of the set of translates and modulates of g:

$$\mathcal{G}(g, \Lambda) = \{e^{2\pi i \langle \lambda, x\rangle} g(x - t) : (t, \lambda) \in \Lambda\}. \tag{1.1}$$

Such systems were first introduced by Gabor [5] who used them for applications in the theory of telecommunication, but there has been a more recent interest in using Gabor system to expand functions both from a theoretical and applied perspective. The branch of Fourier analysis dealing with Gabor systems is usually referred to as Gabor, or time-frequency, analysis. Gröchenig's monograph [6] provide an excellent and detailed exposition on this subject.

Recall that the Gabor system is a *frame* for $L^2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{(t,\lambda)\in\Lambda} |\langle f, e^{2\pi i\langle\lambda,\cdot\rangle} g(\cdot - t)\rangle|^2 \le B\|f\|^2, \quad f \in L^2(\mathbb{R}^d). \tag{1.2}$$

It is called an orthonormal basis for $L^2(\mathbb{R}^d)$ if it is complete and the elements of the system (1.1) are mutually orthogonal in $L^2(\mathbb{R}^d)$ and have norm 1, or, equivalently, $\|g\| = 1$ and $A = B = 1$ in (1.2). One of the fundamental problems in Gabor analysis is to classify the windows g and time-frequency sets Λ with the property that the associated Gabor system $\mathcal{G}(g, \Lambda)$ forms a (Gabor) frame or an orthonormal basis for $L^2(\mathbb{R}^d)$. This is of course a very difficult problem and only partial results are known. For example, to the best of our knowledge, the complete characterization of time-frequency sets Λ for which (1.1) is a frame for $L^2(\mathbb{R}^d)$ was only done when $g = e^{-\pi x^2}$, the Gaussian window. Lyubarskii, and Seip and Wallsten [15,17] showed that $\mathcal{G}(e^{-\pi x^2}, \Lambda)$ is a Gabor frame if and only if the lower Beurling density of Λ is strictly greater than 1. If we assume that Λ is a lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$, then it is well known that $ab \le 1$ is a necessary condition for (1.1) to form a frame for $L^2(\mathbb{R}^d)$. Gröchenig and Stöckler [7] showed that for totally positive functions, (1.1) is a frame if and only if $ab < 1$. If we consider $g = \chi_{[0,c)}$, the characteristic function of an interval, the associated characterization problem is known as the *abc-problem* in Gabor analysis. By rescaling, one may assume that $c = 1$. In that case, the famous Janssen tie showed that the structure of the set of couples (a, b) yielding a frame is very complicated [9,8]. A complete solution of the abc-problem was recently obtained by Dai and Sun [2].

In this paper, we focus our attention on Gabor system of the form (1.1) which yield orthonormal bases for $L^2(\mathbb{R}^d)$. Perhaps the most natural and simplest example of Gabor orthonormal basis is the system $\mathcal{G}(\chi_{[0,1]^d}, \mathbb{Z}^d \times \mathbb{Z}^d)$. The orthonormality property for this

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