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# Markov semigroups with hypocoercive-type generator in infinite dimensions: Ergodicity and smoothing

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A B S T R A C T

We start by considering finite dimensional Markovian dynamics in $\mathbb{R}^m$ generated by operators of hypocoercive type and for such models we obtain short and long time pointwise estimates for all the derivatives, of any order and in any direction, along the semigroup. We then look at infinite dimensional models (in $(\mathbb{R}^m)^{\mathbb{Z}^d}$) produced by the interaction of infinitely many finite dimensional dissipative dynamics of the type indicated above. For these infinite dimensional models we study finite speed of propagation of information, well-posedness of the semigroup, time behaviour of the derivatives and strong ergodicity problem.



## 1. Introduction

In this paper we consider infinite dimensional models of interacting dissipative systems with noncompact state space. In particular we develop a basis for the construction

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and analysis of dissipative semigroups whose generators are given in terms of noncommuting vector fields and for which the equilibrium measures are not a priori known. The ergodicity theory in the case where an invariant measure is not given in advance, in the noncompact subelliptic setup, is an interesting and challenging problem which was initially studied in [9], and which we extend in new directions in this paper, developing a strategy based on generalised gradient bounds. In the following we will first present the main results of the paper and we will then relate them to existing results in the literature.

Hypoelliptic operators of hypocoercive type have received a lot of attention in recent years (see e.g. [12,26] and the references therein), as they naturally arise in non-equilibrium statistical mechanics, for example in the context of the heat bath formalism. These are second order operators on $\mathbb{R}^m$ in Hörmander form

$$L = Z_0^2 + B,$$

where $Z_0$ and $B$ are first order differential operators. The principal part is spanned by at least one field $Z_0$ which, together with the term of first order $B$, generates fields $Z_{j+1} \equiv [B, Z_j]$, $j = 0, \dots, N-1$ spanning the full Lie algebra. Therefore by Hörmander theorem, (see e.g. [13,6,25] and references therein), such semigroups have the strong smoothing property. Motivated by [26], we will refer to these generators as *hypocoercive-type* operators (see Remark 2.2).

At the beginning of the paper, in Section 2, we describe a systematic inductive method which allows to obtain quantitative short and long time estimates for the space-derivatives of the semigroup generated by $L$. We obtain pointwise bounds on the derivatives of any order and in any space direction. The techniques of Section 2 were originally developed in [22] and are based on combining the hypocoercive method presented in [12,26] with the classic Bakry–Emery semigroup approach [2]. Section 2.1 contains an explanation of our technique and its relations with the aforementioned methods in a simplified setting, so that the involved notation of Section 2.2, which is devoted to proving the time behaviour of the derivatives in full generality, does not obfuscate the idea behind the method we present. While obtaining such estimates is an interesting problem in itself, further motivation for obtaining *pointwise* estimates comes from the fact that in the infinite dimensional situation we are interested in this paper, typically one does not have any reference measure. As a consequence, since we do not have the integration by parts trick at our disposal, generally we need to sacrifice estimates in direction of $B$. To the best of our knowledge, a purely analytical method adapted to obtaining pointwise bounds on the time-behaviour of the derivatives of any order of degenerate Markov semigroups was so far lacking. We now come to present the infinite dimensional problem tackled in the subsequent sections of the paper.

Once we have studied the finite dimensional diffusion in $\mathbb{R}^m$ generated by the operator $L$, we study systems of infinitely many interacting diffusions of hypocoercive type. This is done by considering the lattice $\mathbb{Z}^d$ and, roughly speaking, "placing" an isomorphic