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On Gaussian multiplicative chaos

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ABSTRACT

We propose a new definition of the Gaussian multiplicative chaos and an approach based on the relation of subcritical Gaussian multiplicative chaos to randomized shifts of a Gaussian measure. Using this relation we prove general results on uniqueness and convergence for subcritical Gaussian multiplicative chaos that hold for Gaussian fields with arbitrary covariance kernels.

1. Introduction

1.1. The object of interest

Let $(\mathcal{T}, \mu)$ be a finite measure space, and let $X = (X(\omega, t))_{\omega\in\Omega, t\in\mathcal{T}}$ be a Gaussian field parametrized by $t \in \mathcal{T}$ and defined on a probability space (Ω, P). With this data one can associate the following random measure:

$$M(dt) := \exp\left[X(t) - \frac{1}{2}\mathsf{E}\left|X(t)\right|^2\right]\mu(dt). \tag{1}$$

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The Gaussian multiplicative chaos (GMC) is the natural generalization of such a random measure to the setting when the field $(X(t))$ is defined in a distributional sense rather than pointwise, i.e. via a family of formal "integrals" against test functions from an appropriate class. Obviously, for such generalized Gaussian fields (1) does not make sense literally, since $X(t)$ need not be well-defined as a random variable for any particular t. Accordingly, in nontrivial cases M is almost surely μ-singular, so the density $M(dt)/\mu(dt)$ is not well-defined either.

The commonly used ways of interpreting (1) rigorously and constructing such random measures proceed by approximating the field X by Gaussian fields X_n that are, unlike X, defined pointwise. One defines a GMC M as a limit, in an appropriate topology, of random measures

$$M_n(dt) := \exp\left[X_n(t) - \frac{1}{2}\mathsf{E}|X_n(t)|^2\right]\mu(dt). \tag{2}$$

This approach leads naturally to the following problems, both of which will be addressed in this paper.

Problem 1. Find conditions on the approximation $X_n \to X$ that are sufficient for convergence of M_n.

Problem 2. Prove that the limit is independent of the approximation procedure.

As far as we know, in previous works these problems have only been partially solved under unnecessarily restrictive assumptions. Below we provide an overview of the commonly used approximation procedures.

1.1.1. Martingale approximation [11]

The martingale approximation is employed in Kahane's original work on GMC in [11]. In his construction the increments $X_n - X_{n-1}$ are independent (and $X_0 := 0$), which implies that (M_n) is a positive measure-valued martingale. The martingale property guarantees that M_n converges to a random measure M in the sense that $M_n[A] \to M[A]$ almost surely for any fixed measurable set A. Moreover, $\mathsf{E}\,M = \mu$ iff the martingale $(M_n[\mathcal{T}])$ is uniformly integrable, in which case the limit M is taken as the interpretation of (1).

One intuitively expects the martingale approximation to yield the "right" and completely general notion of subcritical GMC. However, Kahane's work falls short of establishing the basic setup in sufficient generality.

The construction in [11] takes as its input a function $K : \mathcal{T} \times \mathcal{T} \to \mathbb{R}_+ \cup \{\infty\}$ — thought of as the covariance kernel of the Gaussian field X. K is assumed to be decomposable into a sum

$$K(t,s) := \sum_n p_n(t,s) \tag{3}$$