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Heat semigroup and singular PDEs

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(with an Appendix by F. Bernicot & D. Frey)

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A B S T R A C T

We provide in this work a semigroup approach to the study of singular PDEs, in the line of the paracontrolled approach developed recently by Gubinelli, Imkeller and Perkowski. Starting from a heat semigroup, we develop a functional calculus and introduce a paraproduct based on the semigroup, for which commutator estimates and Schauder estimates are proved, together with their paracontrolled extensions. This machinery allows us to investigate singular PDEs in potentially unbounded Riemannian manifolds under mild geometric conditions. As an illustration, we study the generalized parabolic Anderson model equation and prove, under mild geometric conditions, its well-posed character in Hölders spaces, in small time on a potentially unbounded 2-dimensional Riemannian manifold, for an equation driven by a weighted noise, and for all times for the linear parabolic Anderson model equation in 2-dimensional unbounded manifolds. This machinery can be extended to an even more

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singular setting and deal with Sobolev scales of spaces rather than Hölder spaces.

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1. Introduction

1.1. State of the art

Following the recent breakthrough of Hairer [37] and Gubinelli, Imkeller, Perkowski [32], there has been recently a tremendous activity in the study of parabolic singular partial differential equations (PDEs), such as the KPZ equation

$$\big(\partial_t - \partial_x^2\big)u = \big(\partial_x u\big)^2 + \xi,$$

the stochastic quantization equation

$$\big(\partial_t - \Delta\big)u = -u^3 + \xi,$$