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Localization principle for compact Hankel operators

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ABSTRACT

In the power scale, the asymptotic behavior of the singular values of a compact Hankel operator is determined by the behavior of the symbol in a neighborhood of its singular support. In this paper, we discuss the localization principle which says that the contributions of disjoint parts of the singular support of the symbol to the asymptotic behavior of the singular values are independent of each other. We apply this principle to Hankel integral operators and to infinite Hankel matrices. In both cases, we describe a wide class of Hankel operators with power-like asymptotics of singular values. The leading term of this asymptotics is found explicitly.

1. Introduction and main results

1.1. Hankel operators on the unit circle

Hankel operators admit various unitarily equivalent descriptions. We start by recalling the definition of Hankel operators on the Hardy class $H^2(\mathbb{T})$. Here $\mathbb{T}$ is the

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unit circle in the complex plane, equipped with the normalized Lebesgue measure $dm(\mu) = (2\pi i\mu)^{-1}d\mu$, $\mu \in \mathbb{T}$; the Hardy class $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ is defined in the standard way as the subspace of $L^2(\mathbb{T})$ spanned by the functions $1, \mu, \mu^2, \dots$. Let $P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the orthogonal projection onto $H^2(\mathbb{T})$, and let W be the involution in $L^2(\mathbb{T})$ defined by $(Wf)(\mu) = f(\bar{\mu})$. For a function $\omega \in L^\infty(\mathbb{T})$, which is called a *symbol* in this context, the *Hankel operator* $H(\omega)$ is defined by the relation

$$H(\omega)f = P_+(\omega W f). \tag{1.1}$$

Background information on the theory of Hankel operators can be found e.g. in the books [6,7].

Recall that the singular values of a compact operator H are defined by the relation $s_n(H) = \lambda_n(|H|)$, where $\{\lambda_n(|H|)\}_{n=1}^\infty$ is the non-increasing sequence of eigenvalues of the compact positive operator $|H| = \sqrt{H^*H}$ (enumerated with multiplicities taken into account). The study of singular values of compact Hankel operators has a long history and is linked to rational approximation, control theory and other subjects, see, e.g. [7]. In fact, this paper is in part motivated by its applications in [12] to the rational approximation of functions with logarithmic singularities.

Singular values $s_n(H(\omega))$ of a Hankel operator with a symbol $\omega \in C^\infty(\mathbb{T})$ decay faster than any power of n^{-1} as $n \to \infty$. On the other hand, the singularities of ω generate a slower decay of singular values. Here we will be interested in the case when the singular values behave as some power of n^{-1}. Optimal upper estimates on singular values of Hankel operators are due to V. Peller, see [7, Section 6.4]. He found necessary and sufficient conditions on ω for the estimate

$$s_n(H(\omega)) \le Cn^{-\alpha}$$

for some $\alpha > 0$. These conditions are stated in terms of the Besov–Lorentz classes.

It is natural to expect that the asymptotic behavior of singular values is determined by the behavior of the symbol ω in a neighborhood of its singular support. We justify this thesis and show that the contributions of the disjoint components of the singular support of ω to the asymptotics of the singular values of $H(\omega)$ are independent of each other. We use the term "localization principle" for this fact. This principle is well understood in the context of the study of the essential spectrum [8] and of the absolutely continuous spectrum [4] of non-compact Hankel operators. Our aim here is to bring this principle to the fore in the question of the asymptotics of singular values of compact Hankel operators.

In our applications the singular support of ω consists of a finite number of points. We use the results of our previous publication [11] (where the history of the problem is described) on the asymptotic behavior of eigenvalues of certain classes of self-adjoint Hankel operators. The localization principle allows us to combine the contributions of different singular points and thus to determine the asymptotics of singular values for a