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Entropy dissipation estimates for the linear Boltzmann operator

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A B S T R A C T

We prove a linear inequality between the entropy and entropy dissipation functionals for the linear Boltzmann operator (with a Maxwellian equilibrium background). This provides a positive answer to the analogue of Cercignani's conjecture for this linear collision operator. Our result covers the physically relevant case of hard-spheres interactions as well as Maxwellian kernels, both with and without a cut-off assumption. For Maxwellian kernels, the proof of the inequality is surprisingly simple and relies on a general estimate of the entropy of the gain operator due to [27,32]. For more general kernels, the proof relies on a comparison principle. Finally, we also show that in the grazing collision limit our results allow to recover known logarithmic Sobolev inequalities.

Contents

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1. Introduction

1.1. Setting of the problem and main result

The use of Lyapunov functionals is a well-known technique to study the asymptotic behavior of dynamical systems, and in the theory of the Boltzmann equation and related models it is now a classical tool. For the nonlinear, spatially homogeneous Boltzmann equation

$$\partial_t f = \mathcal{Q}(f,f), \qquad f(0,v) = f_0(v), \qquad v \in \mathbb{R}^d,\ t \geq 0, \tag{1.1}$$

posed for a function $f = f(t,v)$ depending on $t \geq 0$ and $v \in \mathbb{R}^d$, it is a well-known fact that $f(t,v)$ converges (as $t \to \infty$) towards the Maxwellian distribution M_f with same mass, momentum and energy as f_0,

$$M_f(v) = \frac{\varrho_f}{(2\pi\, E_f)^{d/2}} \exp\left(-\frac{|v-\mathbf{u}_f|^2}{2\,E_f}\right), \qquad v \in \mathbb{R}^d,$$

where

$$\left.\begin{aligned} \varrho_f &= \int_{\mathbb{R}^d} f(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\,\mathrm{d}v, \\ \varrho_f \mathbf{u}_f &= \int_{\mathbb{R}^d} f(t,v) v\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v) v\,\mathrm{d}v, \\ d\,\varrho_f E_f &= \int_{\mathbb{R}^d} f(t,v)|v-\mathbf{u}_f|^2\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\,|v-\mathbf{u}_f|^2\,\mathrm{d}v \end{aligned}\right\} \qquad \text{for all } t \geq 0.$$

Notice that Eq. (1.1) conserves density, momentum and kinetic energy which explains why the above quantities ϱ_f, $\mathbf{u}_f$ and E_f are constant in time. The *Shannon–Boltzmann relative entropy* of f with respect to the Maxwellian distribution M_f