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Adams inequality on the hyperbolic space

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ABSTRACT

In this article we establish the following Adams type inequality in the hyperbolic space $\mathbb{H}^N$:

$$\sup_{u \in C_c^\infty(\mathbb{H}^N),\ \int_{\mathbb{H}^N} (P_k u)u\ dv_g \leq 1} \int_{\mathbb{H}^N} (e^{\beta u^2} - 1)\ dv_g < \infty$$

iff $\beta \leq \beta_0(N,k)$ where $2k = N$, P_k is the critical GJMS operator in $\mathbb{H}^N$ and $\beta_0(N,k)$ is as defined in (1.3). As an application we prove the asymptotic behaviour of the best constants in Sobolev inequalities when $2k = N$ and also prove some existence results for the Q_k curvature type equation in $\mathbb{H}^N$.

1. Introduction

The main focus of this article is on the optimal Adams inequality in space forms. This inequality was established in the zero curvature case $\mathbb{R}^N$ by D.R. Adams [1] and in the constant positive sectional curvature case by Fontana [14]. In this article we establish it

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in the case of hyperbolic space. The inequality we prove (see Theorem 1.1) is in view of the PDE which governs the critical $Q_{\frac{N}{2}}$ curvature under a conformal change of the metric.

Recall the Sobolev embedding theorem which states that if Ω is a bounded domain in $\mathbb{R}^N$, then the Sobolev space $H_0^k(\Omega)$ is continuously embedded in $L^p(\Omega)$ for all $1 \leq p \leq \frac{2N}{N-2k}$, if $2k < N$ and when $2k > N$, $H_0^k(\Omega)$ is continuously embedded in $C^{m,\alpha}(\Omega)$ where $m = k - \left[\frac{N}{2}\right] - 1$ and $\alpha = \left[\frac{N}{2}\right] + 1 - \frac{N}{2}$ if N is odd, otherwise $\alpha \in (0,1)$ is any arbitrary number. One can easily see that when $N = 2k$, neither of the above embeddings are true.

When $k = 1$, an embedding for this case was obtained by Pohožaev [27] and Trudinger [32].

It is well known that the optimal Sobolev embedding plays an important role in several geometric PDEs, like the Yamabe problem, prescribing the scalar curvature, etc. In 1971, J. Moser [24] while trying to study the question of prescribing the Gaussian curvature on the sphere understood the need for establishing a sharp form of the embedding obtained by Pohožaev and Trudinger. He showed that there exists a positive constant C_0 depending only on N such that

$$\sup_{u \in C_c^\infty(\Omega), \int_\Omega |\nabla u|^N \leq 1} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} \, dx \leq C_0 |\Omega| \tag{1.1}$$

holds for all $\alpha \leq \alpha_N = N[\omega_{N-1}]^{\frac{1}{N-1}}$, where Ω is a bounded domain in $\mathbb{R}^N$, and $|\Omega|$ denotes the volume of Ω and ω_{N-1} is the $(N-1)$-dimensional measure of S^{N-1}. Moreover when $\alpha > \alpha_N$, the above supremum is infinite.

In 1988, D.R. Adams [1] established the sharp embedding in the case of higher order Sobolev spaces. He found the sharp constant β_0 for the higher order Trudinger–Moser type inequality. More precisely he proved that if k is a positive integer less than N, then there exists a constant $c_0 = c_0(k, N)$ such that

$$\sup_{u \in C_c^k(\Omega), \int_\Omega |\nabla^k u|^p \leq 1} \int_\Omega e^{\beta |u(x)|^{p'}} \, dx \leq c_0 |\Omega|, \tag{1.2}$$

for all $\beta \leq \beta_0(k, N)$, where $p = \frac{N}{k}, p' = \frac{p}{p-1}$,

$$\beta_0(k,N) = \begin{cases} \dfrac{N}{\omega_{N-1}} \left[\dfrac{\pi^{\frac{N}{2}} 2^k \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{N-k+1}{2}\right)} \right]^{p'}, & \text{if } k \text{ is odd,} \\ \dfrac{N}{\omega_{N-1}} \left[\dfrac{\pi^{\frac{N}{2}} 2^k \Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{N-k}{2}\right)} \right]^{p'}, & \text{if } k \text{ is even,} \end{cases} \tag{1.3}$$

and ∇^k is defined by