Journal of Functional Analysis 270 (2016) 1888–1921

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Atomic and molecular decompositions in variable exponent 2-microlocal spaces and applications ☆

Alexandre Almeida *, António Caetano

Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

A R T I C L E I N F O

Article history:
Received 11 August 2015
Accepted 16 November 2015
Available online 28 November 2015
Communicated by B. Schlein

MSC:
46E35
46E30
42B25
42B08

Keywords:
Variable exponents
Besov–Triebel–Lizorkin spaces
Atoms and Molecules
Sobolev embeddings

A B S T R A C T

In this article we study atomic and molecular decompositions in 2-microlocal Besov and Triebel–Lizorkin spaces with variable integrability. We show that, in most cases, the convergence implied in such decompositions holds not only in the distributions sense, but also in the function spaces themselves. As an application, we give a simple proof for the denseness of the Schwartz class in such spaces. Some other properties, like Sobolev embeddings, are also obtained via atomic representations.

☆ This work was partially supported by Portuguese funds through CIDMA (Center for Research and Development in Mathematics and Applications) and FCT (Foundation for Science and Technology) within project UID/MAT/04106/2013.

* Corresponding author.
E-mail addresses: jaralmeida@ua.pt (A. Almeida), acaetano@ua.pt (A. Caetano).

http://dx.doi.org/10.1016/j.jfa.2015.11.010

1. Introduction

In this paper we deal with atomic and molecular decompositions for 2-microlocal spaces of Besov and Triebel–Lizorkin type with all exponents variable, including applications, for example, to Sobolev type embeddings. This work can be viewed as a continuation of our paper [2], where key properties like characterizations by Peetre maximal functions, liftings and Fourier multipliers were studied.

We also refer to [2] (and the references therein) for a review on the scales $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ helping to contextualize our study. We recall that the investigation of function spaces with variable exponents has been partially motivated by applications to fluid dynamics [34], image processing [7,18,36], PDE and the calculus of variations [1,13,30]; see also the monographs [8,9] and the survey [19] for further details.

Atomic and molecular representations for the spaces $B^{\boldsymbol{w}}_{p(\cdot),q}$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ (so with constant q in the B case) were already obtained by Kempka in [25]. In the present article we give characterizations in terms of atoms and molecules for the full scales above including the difficult case of variable q in the Besov space $B^{w}_{p(\cdot),q(\cdot)}$ (see Section 4) which, as we can see later, is far from being a mere extension. In fact, the mixed sequences spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ behind do not share some fundamental properties as in the constant exponent situation (like the boundedness of the Hardy–Littlewood maximal operator), and they are hard to deal with particularly when the exponent q is unbounded.

We would like to emphasize that even in the cases studied in [25] our statements have different formulations. The idea is to improve and clarify some points and also to give additional information which is hard to find in the standard literature on atomic/molecular representations. By this reason we give some proofs and comments in a separate part (see Section 7).

Another of the main results (see Section 5) asserts that in most cases the convergence implied in the atomic/molecular decompositions holds in the spaces themselves (see Theorem 5.1). We show that this remarkable effect, not usually referred in the literature (but see [6] for an exception in the framework of classical spaces), may have interesting consequences (for example, to immediately get the denseness of the Schwartz class in the spaces $B^{w}_{p(\cdot),q(\cdot)}$ and $F^{w}_{p(\cdot),q(\cdot)}$ when p and q are bounded). Sobolev type embeddings are also established as application of the atomic decompositions obtained (see Section 6).

In Sections 2 and 3 we review some background material and derive some preliminary results needed in the sequel.

2. Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, by $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\mathbb{Z}^n$ we denote the lattice of all points in $\mathbb{R}^n$ with integer components. If r is a real number then $r_+ := \max\{r, 0\}$. We write $B(x, r)$ for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We use c as a generic positive constant, i.e. a constant whose value may change with each appearance. The