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Almost commuting permutations are near commuting permutations ☆

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We prove that the commutator is stable in permutations endowed with the Hamming distance, that is, two permutations that almost commute are near two commuting permutations. Our result extends to k-tuples of almost commuting permutations, for any given k, and allows restrictions, for instance, to even permutations.

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1. Introduction

A famous open problem asks whether or not two almost commuting matrices are necessarily close to two exactly commuting matrices. This is considered independently of the matrix sizes and the terms "almost" and "close" are specified with respect to a given norm. The problem naturally generalizes to k-tuples of almost commuting matrices. It has also a quantifying aspect in estimating the required perturbation and an algorithmic issue in searching for the commuting matrices whenever they do exist.

The current literature on this problem, and its operator and C^*-algebras variants, is immense. The positive answers and counterexamples vary with matrices, matrix norms, and the underlying field, we are interested in. For instance, for self-adjoint complex matrices and the operator norm the problem is due to Halmos [14]. Its affirmative solution for pairs of matrices is a major result of Lin [17], see also [8]. A counterexample for triples of self-adjoint matrices was constructed by Davidson [1] and for pairs of unitary matrices, again with respect to the operator norm, by Voiculescu [24], see also [5]. For the normalized Hilbert–Schmidt norm on complex matrices, the question was explicitly formulated by Rosenthal [22]. Several affirmative and quantitative results for k-tuples of self-adjoint, unitary, and normal matrices with respect to this norm have been obtained recently [12,13,9,6,7].

The problem is also renowned thanks to its connection to physics, originally noticed by von Neumann in his approach to quantum mechanics [20]. The commutator equation being an example, the existence of exactly commuting matrices near almost commuting matrices can be viewed in a wider context of *stability* conceived by Ulam [23, Chapter VI]: an equation is stable if an almost solution (or a solution of the corresponding inequality) is near an exact solution.

Our main result is the *stability of the commutator* in permutations endowed with the normalized Hamming distance, see Definition 2.1 for details on that distance and Definition 3.2 for a precise formulation of the notion of stability.

Main theorem. *For any given $k \geqslant 2$ and with respect to the normalized Hamming distance, every k (even) permutations that almost commute are near k commuting (respectively, even) permutations.*

The interest to the problem on the stability of the commutator in permutations has appeared very recently in the context of sofic groups [10]. Although, permutation matrices are unitary and the Hamming distance can be expressed using the Hilbert–Schmidt distance,[1] the above mentioned techniques available for unitary matrices, equipped with the Hilbert–Schmidt norm, do not provide successful tools towards the stability of the commutator in permutations.

[1] We have $d_H(p,q) = \frac{1}{2} d_{HS}(A_p, A_q)^2$, where $p, q \in Sym(n)$, and A_p, A_q denote the corresponding $n \times n$ permutation matrices, d_H is the Hamming distance, and d_{HS} is the Hilbert–Schmidt distance, both normalized.