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Global regularity for a class of quasi-linear local and nonlocal elliptic equations on extension domains

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ABSTRACT

We consider the solvability of quasi-linear elliptic equations with either local or nonlocal Neumann, Robin, or Wentzell boundary conditions, defined (in the generalized sense) on a bounded $W^{1,p}$-extension domain whose boundary is an upper d-set, for an appropriate $d \geq 0$. Then, we extend the fine regularity theory for weak solutions of the elliptic equations with the above boundary conditions, known for bounded Lipschitz domains, to bounded $W^{1,p}$-extension domains whose boundaries are upper d-sets, by showing that such weak solutions are globally Hölder continuous. Consequently, we generalize substantially the class of bounded domains where weak solutions of boundary value problems of type Neumann, Robin, or Wentzell, may be uniformly continuous (up to the boundary).

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1. Introduction

The main purpose of this article is to extend the global regularity theory for weak solutions of quasi-linear elliptic equations with either Neumann, Robin, or Wentzell boundary conditions (which can be either local or nonlocal), to a large class of bounded domains (open and connected) that include many fractal domains, and domains with non-Lipschitz boundary conditions. Although the boundaries of such domains may be rough and irregular, we point out that boundary value problems on such kinds of regions have been of interest for many specialists in the area, with many applications in probability theory and fractal analysis, among others (e.g. [1,3,5,14], and the references therein). Therefore, any substantial progress in this areas will be very valuable for many authors in different branches of mathematics.

To begin our discussion, for the moment, assume that $\Omega \subseteq \mathbb{R}^N$ is a bounded Lipschitz domain ($N \geq 2$), that is, a bounded domain whose boundary $\Gamma := \partial\Omega$ is locally the graph of a Lipschitz function. Then, given $p \in (1, N)$, $f \in L^r(\Omega, dx)$, and $g \in L^s(\Gamma, d\mathscr{H}^{N-1})$ (for $r, s \in [1, \infty]$ given, and where $\mathscr{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure restricted to Γ, which in this case coincides with the classical surface measure on Γ), we consider the boundary value problems

$$\begin{cases} -\Delta_p u + = f & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = g & \text{on } \Gamma, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \eta|u|^{p-2}u = g & \text{on } \Gamma, \end{cases} \tag{1.2}$$

and

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} - \Delta_{p,\Gamma} u + \eta|u|^{p-2}u = g & \text{on } \Gamma, \end{cases} \tag{1.3}$$

for $\eta \in L^\infty(\Gamma, d\mathscr{H}^{N-1})^+$, where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator, and $\Delta_{p,\Gamma} u := \text{div}(|\nabla_\Gamma u|^{p-2}\nabla_\Gamma u)$ is the so called p-Laplace–Beltrami operator, for ∇_Γ being the tangential gradient at $\Gamma := \partial\Omega$ (see next section for more details). Then Eq. (1.1) represents a Neumann boundary value problem, (1.2) stands as the p-Laplace Eq. with Robin boundary conditions, and problem (1.3) defines a quasi-linear p-Laplace Eq. with (pure) Wentzell boundary conditions. Then, under additional restrictions on r and s, and under our previous assumption (Ω a bounded Lipschitz domain), it was obtained by Nittka [37] that weak solutions of problems (1.1) and (1.2) are Hölder continuous over $\overline{\Omega}$, and the Hölder continuity of Eq. (1.3) was established by Warma [52], in this case under an additional assumption on p.