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Equations involving fractional Laplacian operator: Compactness and application

Shusen Yan [a], Jianfu Yang [b,*], Xiaohui Yu [c]

[a] Department of Mathematics, The University of New England, Armidale, NSW 2351, Australia
[b] Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, PR China
[c] Institute for Advanced Study, Shenzhen University, Shenzhen, Guangdong 518060, PR China

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ABSTRACT

In this paper, we consider the following problem involving fractional Laplacian operator:

$$(-\Delta)^\alpha u = |u|^{2^*_\alpha - 2-\varepsilon} u + \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \qquad (1)$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\varepsilon \in [0, 2^*_\alpha - 2)$, $0 < \alpha < 1$, $2^*_\alpha = \frac{2N}{N-2\alpha}$, and $(-\Delta)^\alpha$ is either the spectral fractional Laplacian or the restricted fractional Laplacian. We show for problem (1) with the spectral fractional Laplacian that for any sequence of solutions u_n of (1) corresponding to $\varepsilon_n \in [0, 2^*_\alpha - 2)$, satisfying $\|u_n\|_H \le C$ in the Sobolev space H defined in (1.2), u_n converges strongly in H provided that $N > 6\alpha$ and $\lambda > 0$. The same argument can also be used to obtain the same result for the restricted fractional Laplacian. An application of this compactness result is that problem (1) possesses infinitely many solutions under the same assumptions.

* Corresponding author.
E-mail addresses: syan@turing.une.edu.au (S. Yan), jfyang_2000@yahoo.com (J. Yang), yuxiao_211@163.com (X. Yu).

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1. Introduction

In this paper, we consider the following problem with the fractional Laplacian:

$$\begin{cases} (-\Delta)^\alpha u = |u|^{2^*_\alpha - 2 - \varepsilon} u + \lambda u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\varepsilon \in [0, 2^*_\alpha - 2)$, $\lambda > 0$, $0 < \alpha < 1$, and $2^*_\alpha = \frac{2N}{N-2\alpha}$ is the critical exponent in fractional Sobolev inequalities.

In a bounded domain $\Omega \subset \mathbb{R}^N$, we define the operator $(-\Delta)^\alpha$ as follows. Let $\{\lambda_k, \varphi_k\}_{k=1}^\infty$ be the eigenvalues and corresponding eigenfunctions of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$ normalized by $\|\varphi_k\|_{L^2(\Omega)} = 1$, i.e.

$$-\Delta \varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega; \quad \varphi_k = 0 \quad \text{on } \partial\Omega.$$

For any $u \in L^2(\Omega)$, we may write

$$u = \sum_{k=1}^\infty u_k \varphi_k, \quad \text{where} \quad u_k = \int_\Omega u \varphi_k \, dx.$$

We define the space

$$H = \{u = \sum_{k=1}^\infty u_k \varphi_k \in L^2(\Omega) : \sum_{k=1}^\infty \lambda_k^\alpha u_k^2 < \infty\}, \tag{1.2}$$

which is equipped with the norm

$$\|u\|_H = \left(\sum_{k=1}^\infty \lambda_k^\alpha u_k^2 \right)^{\frac{1}{2}}.$$

For any $u \in H$, the spectral fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u = \sum_{k=1}^\infty \lambda_k^\alpha u_k \varphi_k.$$

With this definition, we see that problem (1.1) with $\varepsilon = 0$ is the Brézis–Nirenberg type problem with the fractional Laplacian. In [6], Brézis and Nirenberg considered the existence of positive solutions for problem (1.1) with $\alpha = 1$ and $\varepsilon = 0$. Such a problem involves the critical Sobolev exponent $2^* = \frac{2N}{N-2}$ for $N \geq 3$, and it is well known that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact even if Ω is bounded. Hence, the associated functional of problem (1.1) does not satisfy the Palais–Smale condition, and