Journal of Functional Analysis 269 (2015) 80–154

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Wavelet coorbit spaces viewed as decomposition spaces

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A R T I C L E   I N F O

Article history:
Received 23 June 2014
Accepted 27 March 2015
Available online 27 April 2015
Communicated by P. Biane

MSC:
42B35
42C40
46F05

Keywords:
Coorbit spaces
Decomposition spaces
Function spaces
Anisotropic wavelet systems

A B S T R A C T

In this paper we show that the Fourier transform induces an isomorphism between the coorbit spaces defined by Feichtinger and Gröchenig of the mixed, weighted Lebesgue spaces $L_v^{p,q}$ with respect to the quasi-regular representation of a semi-direct product $\mathbb{R}^d \rtimes H$ with suitably chosen dilation group H, and certain decomposition spaces $\mathcal{D}(\mathcal{Q}, L^p, \ell_u^q)$ (essentially as introduced by Feichtinger and Gröbner) where the localized "parts" of a function are measured in the $\mathcal{F}L^p$-norm.

This equivalence is useful in several ways: It provides access to a Fourier-analytic understanding of wavelet coorbit spaces, and it allows to discuss coorbit spaces associated to different dilation groups in a common framework. As an illustration of these points, we include a short discussion of dilation invariance properties of coorbit spaces associated to different types of dilation groups.

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http://dx.doi.org/10.1016/j.jfa.2015.03.019

1. Introduction

There exist several methods in the literature for the construction of higher-dimensional wavelet systems. A rather general class of constructions follows the initial inception of the continuous wavelet transform in [21] and uses the language of group representations [26,1,16,24]: Picking a suitable matrix group $H \leq \mathrm{GL}\left(\mathbb{R}^d\right)$, the **dilation group**, one defines the associated semidirect product $G = \mathbb{R}^d \rtimes H$. This group acts on $\mathrm{L}^2(\mathbb{R}^d)$ via the (unitary) **quasi-regular representation** π defined by

$$(\pi(x,h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y-x)) \ , \quad (x,h) \in \mathbb{R}^d \times H.$$

The associated continuous wavelet transform of a signal $f \in \mathrm{L}^2(\mathbb{R}^d)$ is then obtained by picking a suitable mother wavelet $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, and letting

$$W_\psi f : G \to \mathbb{C} \ , \ (x,h) \mapsto \langle f, \pi(x,h)\psi \rangle \ . \tag{1.1}$$

A wavelet ψ is called **admissible** if the operator W_ψ is (a multiple of) an isometry as a map into $\mathrm{L}^2(G, \mu_G)$, where μ_G denotes a left Haar measure on G. By definition we thus have for admissible vectors ψ that

$$\forall f \in \mathrm{L}^2(\mathbb{R}^d) \ : \ \|f\|_2^2 = \frac{1}{C_\psi} \cdot \int_H \int_{\mathbb{R}^d} |W_\psi f(x,h)|^2 \, \mathrm{d}x \, \frac{\mathrm{d}h}{|\det(h)|} \ ,$$

alternatively expressed in the weak-sense inversion formula

$$f = \frac{1}{C_\psi} \cdot \int_H \int_{\mathbb{R}^d} W_\psi f(x,h) \cdot \pi(x,h)\psi \ \mathrm{d}x \ \frac{\mathrm{d}h}{|\det(h)|} \ .$$

An alternative construction of wavelet systems, with somewhat less structure but higher design flexibility, is the semi-discrete approach described as follows: Pick a discretely labeled quadratic partition of unity $(\widehat{\psi_i})_{i \in I}$ in frequency domain, i.e. a family of functions satisfying

$$\forall_{a.e.} \xi \in \mathbb{R}^d \ : \ \sum_{i \in I} \left| \widehat{\psi_i}(\xi) \right|^2 = 1 \tag{1.2}$$

and consider the system of all translates of the inverse Fourier transforms $\psi_i = \mathcal{F}^{-1}(\widehat{\psi_i})$. This system is a (continuously labeled) tight frame, i.e.

$$\forall f \in \mathrm{L}^2(\mathbb{R}^d) \ : \ \|f\|_2^2 = \sum_{i \in I} \int_{\mathbb{R}^d} |\langle f, L_x \psi_i \rangle|^2 \, \mathrm{d}x,$$