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On Kirchberg's embedding problem ☆

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ABSTRACT

Kirchberg's Embedding Problem (KEP) asks whether every separable C* algebra embeds into an ultrapower of the Cuntz algebra $\mathcal{O}_2$. In this paper, we use model theory to show that this conjecture is equivalent to a local approximate nuclearity condition that we call *the existence of good nuclear witnesses*. In order to prove this result, we study general properties of existentially closed C* algebras. Along the way, we establish a connection between existentially closed C* algebras, the weak expectation property of Lance, and the local lifting property of Kirchberg. The paper concludes with a discussion of the model theory of $\mathcal{O}_2$. Several results in this last section are proven using some technical results concerning tubular embeddings, a notion first introduced by Jung for studying embeddings of tracial von Neumann algebras into the ultrapower of the hyperfinite II_1 factor.

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Contents

1. Introduction

The Cuntz algebra $\mathcal{O}_2$ is defined to be the universal C*-algebra generated by two isometries v_1, v_2 satisfying the relation $v_1v_1^* + v_2v_2^* = 1$. It is a unital, separable, simple, nuclear, purely infinite C*-algebra (we refer to such algebras in short as *Kirchberg algebras*). In the 1990s Kirchberg obtained two remarkable theorems which established $\mathcal{O}_2$ as a central object of study in the K-theoretic classification of separable, nuclear C*-algebras: $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ for any Kirchberg algebra A; and (jointly with N. Phillips) that any separable, exact (e.g., nuclear) C*-algebra is embeddable in $\mathcal{O}_2$ (this in fact characterizes exactness).

The techniques developed by Kirchberg and Kirchberg–Phillips to prove the $\mathcal{O}_2$ embedding theorem in fact suggest the possibility that *every* separable C*-algebra is embeddable into an ultrapower of $\mathcal{O}_2$, though there are many examples of non-exact, separable C*-algebras also due to Kirchberg [22]. This possibility was already noticed by Kirchberg and we refer to the question of whether every separable C*-algebra is $\mathcal{O}_2^\omega$-embeddable as *Kirchberg's Embedding Problem* (KEP). We remark that this problem should not be conflated with Kirchberg's conjecture that there is a unique C*-norm on the algebraic tensor product $C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty)$, which is known to be equivalent to Connes' Embedding Problem [22].

The purpose of this paper is to investigate KEP from the perspective of the continuous model theory of C*-algebras. The development of the model theory of C*-algebras in general, and nuclear (or exact) C*-algebras specifically, is in its very early stages, so many results obtained in this work are aimed at developing the foundational aspects of this theory. The results and ideas detailed below are complementary to (and borrow from) a larger systematic treatment of the model theory of nuclear C*-algebras by Farah et al. [11] which is forthcoming. One important feature of our approach is that we emphasize