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Discrete components in restriction of unitary representations of rank one semisimple Lie groups ☆

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ABSTRACT

We consider spherical principal series representations of the semisimple Lie group of rank one $G = SO(n,1;\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. There is a family of unitarizable representations π_ν of G for ν in an interval on $\mathbb{R}$, the so-called complementary series, and subquotients or subrepresentations of G for ν being negative integers. We consider the restriction of (π_ν, G) under the subgroup $H = SO(n-1,1;\mathbb{K})$. We prove the appearing of discrete components. The corresponding results for the exceptional Lie group $F_{4(-20)}$ and its subgroup $Spin_0(8,1)$ are also obtained.

1. Introduction

The study of direct components in the restriction to a subgroup $H \subset G$ of a representation (π, G) is one of major subjects in representation theory. Among representations of a semisimple Lie group G there are two somewhat opposite classes, the discrete series and the complementary series; the former appear in the decomposition of $L^2(G)$ and can be treated algebraically, whereas the latter do not contribute to the decomposition and

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their study involves more analytic issues. The study of restriction of discrete series representations has been studied intensively; see e.g. [18,28] and references therein. Motivated by some related questions of [2,3] Speh and Venkataramana [30] studied the restriction of a complementary series representation of $SO(n,1)$ under the subgroup $SO(n-1,1)$. It is proved there, for relatively small parameter ν (in our parametrization), the complementary series π_ν of $SO(n-1,1)$ appears discretely in the complementary series π_ν of $SO(n,1)$ with the same parameter ν. They construct the imbedding of the complementary series of $SO(n-1,1)$ into π_ν of $SO(n,1)$ by using non-compact realizations of the representations as spaces of distributions on Euclidean spaces and by extending distributions on $\mathbb{R}^{n-2}$ to $\mathbb{R}^{n-1}$. Similar results are also obtained for complementary series of differential forms.

In the present paper we shall study the restriction, also called branching, of complementary series of G for all rank one Lie groups G with respect to a symmetric pair (G,H). More precisely we prove the appearance of discrete components for $G = SO(n,1;\mathbb{K})$, $H = SO(n-1,1;\mathbb{K})$, with $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$ being the fields of real, complex, quaternion numbers, or for $G = F_{4(-20)}$ and $H = Spin_0(8,1) \subset G$. We shall use the compact realization of the spherical principal series π_ν on the sphere $S = K/M$ in $\mathbb{F}^n$. We prove that for appropriate small parameter ν the natural restriction map of functions on S in π_ν to the lower dimensional sphere $S^\flat$ in $\mathbb{F}^{n-1}$ defines a bounded operator onto a complementary series $\pi_\nu^\flat$ of H. The proof requires rather detailed study of the restriction to $S^\flat \subset S$ of spherical harmonics on S.

The representations π_ν for certain integers ν have also unitarizable subquotients or subrepresentations; some of them are discrete series representations of G. We shall find also irreducible components for these representations under the subgroup H. One easiest case is the subrepresentation $\pi_0^\pm$ (or $\pi_{2n+2}^\pm$ as quotient) of the group $SU(n,1)$. The space $\pi_0^\pm$ consists of holomorphic respectively antiholomorphic polynomials on $\mathbb{C}^n$ modulo constant functions. It can also be treated by using the analytic continuation of scalar holomorphic discrete series at the reducible point [8], and some general decomposition results have been obtained in [19].

The main results in this paper are summarized in the following theorem, the precise statements being given in Theorems 3.6, 3.9 and 4.4; the parametrization of the complementary series (G, π_ν) is done so that the unitary principal series of G appear for $\nu = \rho_G + it$, $t \in \mathbb{R}$, so that the complementary series appear for ν in a symmetric interval around ρ_G.

Theorem 1.1. *Let (G,H) be the pair as above, $G = SO(n,1;\mathbb{K})$, $H = SO(n-1,1;\mathbb{K})$ for $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$, or $G = F_{4(-20)}$, and $H = Spin_0(8,1) \subset G$. Let $\rho_G = d - 1 + \frac{d}{2}(n-1)$ and $\rho_H = d - 1 + \frac{d}{2}(n-2)$ be the corresponding half sums of positive roots, where $d = \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4$. Suppose (π_ν, G) is a complementary series representation of G. We can assume up to Weyl group symmetry that $\nu < \rho_G$.*

(1) *The restriction of (π_ν, G) on H contains a discrete component $(\pi_\mu^\flat, H)$ if $\nu < \rho_H$, and $\mu = \nu$ in our parameterization.*