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Geometric characterizations of asymptotic flatness and linear momentum in general relativity

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A R T I C L E   I N F O

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A B S T R A C T

In 1996, Huisken–Yau proved that every three-dimensional Riemannian manifold can be uniquely foliated near infinity by stable closed surfaces of constant mean curvature (CMC) if it is asymptotically equal to the (spatial) Schwarzschild solution. Their decay assumptions were weakened by Metzger, Huang, Eichmair–Metzger, and the author at a later stage. In this work, we prove the reverse implication, i.e. any three-dimensional Riemannian manifold is asymptotically flat if it possesses a CMC-*cover* satisfying certain geometric curvature estimates, a uniqueness property, a *weak* foliation property, while each surface has weakly controlled instability. With the author's previous result that every asymptotically flat manifold possesses a CMC-foliation, we conclude that asymptotic flatness is characterized by existence of such a CMC-cover. Additionally, we use this characterization to provide a geometric (i.e. coordinate-free) definition of a *(CMC-)linear momentum* and prove its compatibility with the linear momentum defined by Arnowitt–Deser–Misner.

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1. Introduction

Surfaces of constant mean curvature (CMC) were first used in mathematical general relativity by Christodoulou–Yau who studied quasi-local mass of asymptotically flat manifolds [12]. In 1996, Huisken–Yau proved the existence of a unique foliation by stable CMC-surfaces [22]. They considered Riemannian manifolds $(\overline{\mathrm{M}}, \overline{g}, \overline{x})$ which are asymptotically equal to the (spatial) Schwarzschild solution by assuming the existence of a coordinate system $\overline{x} : \overline{\mathrm{M}} \setminus \overline{L} \to \mathbb{R}^3 \setminus \overline{B_1(0)}$ mapping the manifold (outside of some compact set $\overline{L}$) to the Euclidean space (outside the closed unit ball) such that the push forward $\overline{x}_*\overline{g}$ of the metric $\overline{g}$ is asymptotically equal to the (spatial) Schwarzschild metric. More precisely, they assumed that the decay of the k-th derivatives of the difference $\overline{g}_{ij} - {}^{S}\overline{g}_{ij}$ of the metric $\overline{g}$ and the Schwarzschild metric ${}^{S}\overline{g} := (1 + {\overline{m}}/{2|x|})^4\, {}^{e}\overline{g}$ in these coordinates followed $|\overline{x}|^{-2-k}$ for every $k \leq 4$. Here, the mass $\overline{m}$ was assumed to be positive and ${}^{e}\overline{g}$ denotes the Euclidean metric. This is abbreviated by $\overline{g} = {}^{S}\overline{g} + \mathcal{O}_4(|x|^{-2})$. Later, these decay assumptions were weakened by Metzger, Huang, Eichmair–Metzger, and the author [23,21,17,27]: It is sufficient to assume asymptotic flatness to ensure the existence of a CMC-foliation (and its uniqueness in a well-defined class of surfaces). The metric being asymptotically flat means $\overline{g} = {}^{e}\overline{g} + \mathcal{O}_2(|x|^{-\frac{1}{2}-\varepsilon})$ with $\overline{\mathcal{S}} = \mathcal{O}_0(|x|^{-3-\varepsilon})$, where $\overline{\mathcal{S}}$ the scalar curvature of $\overline{g}$. Further properties of this foliation were studied by Huisken–Yau, Corvino–Wu, Eichmair–Metzger, the author, and others [22,15,17,25,27].

Inspired by an idea by Huisken to use the CMC-foliation to define a unique[1] coordinate system $\overline{y} : \overline{\mathrm{M}} \setminus \overline{K} \to (\sigma_0\,;\infty) \times \mathbb{S}^2$, we prove that asymptotic flatness does not only *imply* the existence and (local) uniqueness of a CMC-*cover* $\{{}_{\sigma}\Sigma\}_{\sigma\geq\sigma_0}$, but is *characterized* by it. A simple form of the more general versions (see Section 4) is the following:

Corollary 1.1 *(Simple version of Theorem 4.1). Let $(\overline{\mathrm{M}}, \overline{g})$ be a three-dimensional Riemannian manifold without boundary and $\varepsilon \in (0\,;\frac{1}{2})$ be a constant. There exists a coordinate system $\overline{x} : \overline{\mathrm{M}} \setminus \overline{L} \to \mathbb{R}^3 \setminus \overline{B_1(0)}$ outside a compact set $\overline{L} \subseteq \overline{\mathrm{M}}$ such that $(\overline{\mathrm{M}}, \overline{g}, \overline{x})$ is $\mathrm{C}^2_{\frac{1}{2}+\varepsilon}$-asymptotically flat with strictly positive ADM-mass if and only if there are constants $c > 0$, $M > 0$, $\sigma_0 \geq \sigma_0' = \sigma_0'(\varepsilon, c, M)$, and a family $\mathcal{M} := \{{}_{\sigma}\Sigma\}_{\sigma>\sigma_0'}$ of spheres in $\overline{\mathrm{M}}$ such that*

(a) *$\mathcal{M}$ is* locally unique*, i.e. if $\Sigma' \hookrightarrow \overline{\mathrm{M}}$ is a CMC-surface and a graph on $\Sigma \in \mathcal{M}$ which is $\mathrm{W}^{2,p}$-close to Σ, then $\Sigma' \in \mathcal{M}$;*
(b) *$\overline{\mathrm{M}} \setminus \bigcup_{\sigma} {}_{\sigma}\Sigma$ is relatively compact;*
(c) *${}_{\sigma}\Sigma$ is stable as surface of constant mean curvature ${}_{\sigma}\mathcal{H} \equiv {-2}/{\sigma}$ for every $\sigma > \sigma_0$;*
(d) *the Hawking masses $m_{\mathrm{H}}({}_{\sigma}\Sigma)$ of the elements ${}_{\sigma}\Sigma \in \mathcal{M}$ are bounded away from zero and infinity, i.e. $|m_{\mathrm{H}}({}_{\sigma}\Sigma)| \in (M^{-1}\,; M)$ for every $\sigma > \sigma_0$;*
(e) *$\sup_{{}_{\sigma}\Sigma} (\sigma^{\frac{5}{2}+\varepsilon}\, |\overline{\mathrm{Ric}}|_{\overline{g}} + \sigma^{3+\varepsilon}\, |\overline{\mathcal{S}}|) \leq c$ for every $\sigma > \sigma_0$;*
(f) *the elements of $\mathcal{M}$ are pairwise disjoint.*

In this setting, $\mathcal{M}$ is a smooth foliation and the ADM-mass $\overline{m}$ of $(\overline{\mathrm{M}}, \overline{g})$ satisfies $\overline{m} = \lim_{\sigma\to\infty} m_{\mathrm{H}}({}_{\sigma}\Sigma)$.

[1] up to Euclidean isometries