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Commuting Toeplitz operators on bounded symmetric domains and multiplicity-free restrictions of holomorphic discrete series

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A B S T R A C T

For any given bounded symmetric domain, we prove the existence of commutative C^*-algebras generated by Toeplitz operators acting on any weighted Bergman space. The symbols of the Toeplitz operators that generate such algebras are defined by essentially bounded functions invariant under suitable subgroups of the group of biholomorphisms of the domain. These subgroups include the maximal compact groups of biholomorphisms. We prove the commutativity of the Toeplitz operators by considering the Bergman spaces as the underlying space of the holomorphic discrete series and then applying known multiplicity-free results for restrictions to certain subgroups of the holomorphic discrete series. In the compact case we completely characterize the subgroups that define invariant symbols that yield commuting Toeplitz operators in terms of the multiplicity-free property.

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1. Introduction

The weighted Bergman spaces on bounded symmetric domains are a fundamental object in Analysis. They come equipped with a natural projection, the Bergman projection, determined by a reproducing kernel property. This structure allows to consider the so-called Toeplitz operators defined as a multiplier operator followed by the Bergman projection. The special role of Toeplitz operators is observed, for example, in their density in the space of all bounded operators in the strong operator topology (see [5]). It is thus a surprising fact that there are large commutative C^*-algebras generated by Toeplitz operators. The existence of such commutative algebras is a current topic of interest in Complex Analysis, as the references of this work show.

The commutative C^*-algebras generated by Toeplitz operators known to this date always have an associated distinguished geometry. The latter is given by a subgroup H of the group of biholomorphic maps of the domain. More precisely, for certain choices of H the H-invariant essentially bounded functions determine commuting Toeplitz operators. Up to this date this kind of phenomenon has been observed only for the unit ball $\mathbb{B}^n$ with H a maximal Abelian subgroup of its biholomorphisms (see [11,24,25]) and some variations (see [2,22,23]) as well as for the natural translations of a tube type domain in the weightless case (see [27]). It is an open problem to find higher rank irreducible bounded symmetric domains that admit for any weight large families of commuting Toeplitz operators acting on their Bergman spaces.

On the other hand, the weighted Bergman spaces are also very important in Harmonic Analysis. In this setup, the Bergman spaces provide the underlying spaces of the holomorphic discrete series representations for noncompact simple Lie groups associated to bounded symmetric domains. The study of such representations is fundamental as part of the picture to understand the unitary representations of simple Lie groups. In particular, a very great deal of attention has been given to the holomorphic discrete series; see for example [7,12,29] just to mention a few.

The representations in the holomorphic discrete series are all irreducible for the action of the whole group of biholomorphisms of the corresponding bounded symmetric domain. However, for a (proper closed) subgroup H of all biholomorphisms the irreducibility is lost to be replaced by a direct integral decomposition into classes of irreducible unitary representations of H. The study of the branching behavior for these representations and many other is a fundamental part of Harmonic Analysis. In particular, for the holomorphic discrete series representations there are very general results that provide some of the subgroups H for which the restricted representation is multiplicity-free, i.e. so that the classes of irreducible representations over H appear with multiplicity 1 in the direct integral decomposition. Some of the results of this sort most relevant to this work can be found in [1,4,18,21].

The main goal of this work is to prove that subgroups defining multiplicity-free restrictions of the holomorphic discrete series yield commutative C^*-algebras generated by Toeplitz operators. To be more precise, we obtain commuting Toeplitz operators when