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Real dimensional spaces in noncommutative geometry

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ABSTRACT

In this paper we will extend the product of spectral triples to a product of semifinite spectral triples. We will prove that finite summability and regularity are preserved under taking products. Connes and Marcolli constructed for each $z \in (0, \infty)$ a type II_∞-semifinite spectral triple which can be considered as a geometric space of dimension z. A small adaption of their construction yields a type I-semifinite spectral triple. The properties of these semifinite spectral triples will be investigated. At the same time we will also avoid the need for an infra-red cutoff to compute the dimension spectrum. An application of these semifinite spectral triples to dimensional regularization in quantum field theory is given.

1. Introduction

In [18] 't Hooft and Veltman developed the method of dimensional regularization to deal with divergent integrals in quantum field theory. The idea they had was to evaluate the corresponding integrals in $d - w$ dimensions for $w \in \mathbb{C}$ instead of the original d dimensions. This approach plays a key role in modern quantum field theory computations. It is therefore a natural question whether it is possible to mathematically construct geometric spaces which have dimension $z \in \mathbb{C}$. As described by Connes and

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Marcolli in [8] this is indeed possible in the framework of noncommutative geometry [5]. They found a spectral triple for which the 'Dirac operator' D satisfies the following generalization of the Gaussian integral in z dimensions:

$$\mathrm{Tr}\left(e^{-\lambda D^2}\right) = \left(\frac{\pi}{\lambda}\right)^{z/2}.$$

Such spectral triples can be found in the generalization of spectral triples to semifinite spectral triples, introduced by Benameur and Fack in the papers [1] and [2].

In the second section of this paper we will focus on these semifinite spectral triples. We recall the standard definitions and shortly discuss the 'dimension spectrum' [9], a subset of $\mathbb{C}$ which gives a notion of dimension to (semifinite) spectral triples. Then we will construct the product of semifinite spectral triples, this is a natural extension of the product of ordinary spectral triples. It will be shown that this product preserves finite summability and regularity. In the third part of this paper we will give the construction of the z-dimensional semifinite spectral triples. We will elaborate on why these can be considered as a generalization of a geometric space of dimension z and we will compare our construction with the definition of Connes and Marcolli. Connes and Moscovici [9] use the operator $|D|^{-1}$ in their definition of the dimension spectrum. This causes some problems, because the Dirac operator is not invertible in the triple we will consider. One way of solving that problem is by imposing an infra-red cutoff. Another solution uses an alternative definition of the 'dimension spectrum'. In this definition the operator $(D^2+1)^{-1/2}$ is used, which in this case is well-defined. However this second definition has as a disadvantage that computations become a lot more complicated. Finally we will show that these aforementioned semifinite spectral triples can indeed be used to define integrals in z dimensions so that we obtain a more concrete picture of dimensional regularization. We will not build a full theory, but we will work out an example in details which illustrates the main ideas. Appendix A is added in which the required results about traces on semifinite von Neumann algebras are stated.

2. Semifinite noncommutative geometry

The objective of Section 3 is to construct spectral triples which satisfy a specific requirement so that they can be considered to be z-dimensional for $z \in (0, \infty)$. The construction which is given makes use of semifinite traces and not the ordinary trace Tr. This naturally leads to the notion of semifinite spectral triples. In this section we will derive some general results about these semifinite spectral triples, in particular we will be concerned with products of such triples.

2.1. Semifinite spectral triples and their properties

The difference between an ordinary spectral triple and a semifinite one is that we no longer require that the resolvent of the Dirac operator is compact, but we want it to be compact relative to a trace on a semifinite von Neumann algebra.