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Essential normality, essential norms and hyperrigidity ☆

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A B S T R A C T

Let $S = (S_1, \dots, S_d)$ denote the compression of the d-shift to the complement of a homogeneous ideal I of $\mathbb{C}[z_1, \dots, z_d]$. Arveson conjectured that S is essentially normal. In this paper, we establish new results supporting this conjecture, and connect the notion of essential normality to the theory of the C*-envelope and the noncommutative Choquet boundary. The unital norm closed algebra $\mathcal{B}_I$ generated by $S_1, \dots, S_d$ modulo the compact operators is shown to be completely isometrically isomorphic to the uniform algebra generated by polynomials on $\overline{V} := \overline{\mathcal{Z}(I) \cap \mathbb{B}_d}$, where $\mathcal{Z}(I)$ is the variety corresponding to I. Consequently, the essential norm of an element in $\mathcal{B}_I$ is equal to the sup norm of its Gelfand transform, and the C*-envelope of $\mathcal{B}_I$ is identified as the algebra of continuous functions on $\overline{V} \cap \partial\mathbb{B}_d$, which means it is a complete invariant of the topology of the variety determined by I in the ball.

Motivated by this determination of the C*-envelope of $\mathcal{B}_I$, we suggest a new, more qualitative approach to the problem of essential normality. We prove the tuple S is essentially normal if and only if it is hyperrigid as the generating set of a C*-algebra, which is a property closely connected to Arveson's notion of a boundary representation.

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We show that most of our results hold in a much more general setting. In particular, for most of our results, the ideal I can be replaced by an arbitrary (not necessarily homogeneous) invariant subspace of the d-shift.

1. Introduction, notation and preliminaries

The purpose of this paper is to collect evidence supporting Arveson's conjecture on essential normality, and to connect the conjecture with the theory of the C*-envelope and the noncommutative Choquet boundary. Our results are of a nature quite different from other results on this conjecture, e.g., [4,5,14–17,19,21,25,26,34]; these previous results gave a full verification of the conjecture for limited classes of (typically homogeneous) ideals. Here, we shall present more limited results that hold for all homogeneous ideals, and for a large class of non-homogeneous ideals. Arveson's conjecture has several interesting and non-trivial consequences. We shall prove some of these consequences directly, thereby gathering evidence supporting the conjecture.

This work also connects to the ongoing effort to understand operator algebras arising from *subproduct systems* (see [12,13,22,23,36–38]). If we restrict attention to homogeneous ideals, then the algebras studied in this paper are precisely the algebras arising from commutative subproduct systems over $\mathbb{N}$, with finite dimensional Hilbert spaces as fibers.

1.1. Preliminaries

Throughout, $d \geq 2$ is a fixed integer, $\mathbb{B}_d$ denotes the unit ball in $\mathbb{C}^d$, and $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_d]$ denotes the algebra of complex polynomials in d variables. Let E be a Hilbert space with orthonormal basis $\{e_1, \dots, e_d\}$. Then we may identify the symmetric tensor algebra over E with $\mathbb{C}[z]$. Form the symmetric Fock space over E:

$$\mathcal{F}^+(E) = \mathbb{C} \oplus E \oplus E^2 \oplus \dots$$

The space $\mathcal{F}^+(E)$ is also called the Drury–Arveson space. It can be naturally identified as the reproducing kernel Hilbert space on the unit ball with reproducing kernel

$$k_w(z) = \frac{1}{1 - \langle z, w \rangle}, \quad w, z \in \mathbb{B}_d.$$

In this function-theoretic incarnation the Drury–Arveson space is usually denoted by H^2_d, and we shall use this notation here. A third, equivalent, way of viewing this space is simply as the completion of $\mathbb{C}[z]$ under the inner product that makes monomials orthogonal and assigns to each monomial the norm