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Sign-changing blow-up solutions for Hénon type elliptic equations with exponential nonlinearity ☆

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ABSTRACT

We study the existence of sign-changing solutions with multiple concentration to the following boundary value problem

$$-\Delta u = \varepsilon^2 |x|^{2\alpha}(e^u - e^{-u}) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\alpha > 0$, Ω is a smooth bounded domain in $\mathbb{R}^2$ containing the origin, $\varepsilon > 0$ is a small parameter. In particular we prove that if $\alpha \neq 1$ then a nodal solution exists with a number of mixed positive and negative blow-up points up to 4.

1. Introduction

Let Ω be a smooth and bounded domain in $\mathbb{R}^2$ with $0 \in \Omega$. In this paper we are concerned with the existence and the asymptotic analysis when the parameter ε tends to 0 of solutions for the following Hénon type elliptic problem:

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$$\begin{cases} -\Delta u = \varepsilon^2 |x|^{2\alpha}(e^u - e^{-u}) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{1.1}$$

Problem (1.1) together with the Liouville model

$$\begin{cases} -\Delta u = \varepsilon^2 |x|^{2\alpha} e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

is motivated by its links with the modeling of physical phenomenon. In particular, (1.1) and (1.2) arise in the study of vortices in a planar model of Euler flows (see [3,11]). In vortex theory the interest in constructing *blowing-up* solutions is related to relevant physical properties, in particular the presence of vortices with a strongly localized electromagnetic field.

The autonomous case, i.e. when $\alpha = 0$, has been widely considered in literature. The asymptotic behavior of blowing-up family of solutions for the problem (1.2) can be referred to the papers [1,7,21–23,25]. More precisely, the analysis in these works yields that if u_ε is an unbounded family of solutions of (1.2) with $\alpha = 0$ for which $\varepsilon^2 \int_\Omega e^{u_\varepsilon}$ is uniformly bounded, then there is an integer $N \geq 1$ such that

$$\varepsilon^2 \int_\Omega e^{u_\varepsilon} dx \to 8\pi N \quad \text{as } \varepsilon \to 0.$$

Moreover there are different points $\xi_1^\varepsilon, \dots, \xi_N^\varepsilon \in \Omega$, which remain uniformly distant from the boundary $\partial\Omega$ and from one another such that

$$\varepsilon^2 e^{u_\varepsilon} - 8\pi \sum_{i=1}^N \delta_{\xi_i^\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \tag{1.3}$$

in the measure sense. Here δ_ξ denotes Dirac mass supported at ξ. Also the location of the blowing-up points is well understood. Indeed, in [23] and [25] it is established that the N-tuple $(\xi_1^\varepsilon, \dots, \xi_N^\varepsilon)$ converges, up to a subsequence, to a critical point of the functional

$$\frac{1}{2} \sum_{i=1}^N H(\xi_i, \xi_i) + \sum_{\substack{i,j=1 \\ i<j}}^N G(\xi_i, \xi_j). \tag{1.4}$$

Here $G(x, y)$ is the Green's function of $-\Delta$ over Ω under Dirichlet boundary conditions, i.e. G satisfies

$$\begin{cases} -\Delta_y G(x, y) = \delta_x(y) & y \in \Omega \\ G(x, y) = 0 & y \in \partial\Omega \end{cases},$$