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Unitary groups and spectral sets

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ABSTRACT

We study spectral theory for bounded Borel subsets of $\mathbb{R}$ and in particular finite unions of intervals. For Hilbert space, we take L^2 of the union of the intervals. This yields a boundary value problem arising from the minimal operator $\mathsf{D} = \frac{1}{2\pi i}\frac{d}{dx}$ with domain consisting of C^∞ functions vanishing at the endpoints. We offer a detailed interplay between geometric configurations of unions of intervals and a spectral theory for the corresponding self-adjoint extensions of D and for the associated unitary groups of local translations. While motivated by scattering theory and quantum graphs, our present focus is on the Fuglede-spectral pair problem. Stated more generally, this problem asks for a determination of those bounded Borel sets Ω in $\mathbb{R}^k$ such that $L^2(\Omega)$ has an orthogonal basis of Fourier frequencies (spectrum), i.e., a total set of orthogonal complex exponentials restricted to Ω. In the general case, we characterize Borel sets Ω having this spectral property in terms of a unitary representation of $(\mathbb{R}, +)$ acting by local translations. The case of $k = 1$ is of special interest, hence the interval-configurations. We give a characterization of those geometric interval-configurations which allow Fourier spectra directly in terms of the self-adjoint extensions of the minimal operator D. This allows for a direct and explicit interplay between geometry and spectra.

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Contents

1. Introduction

In this paper we study a classification problem (SAE) for self-adjoint extensions of Hermitian operators in Hilbert space, with dense domain and finite deficiency indices (n, n). While this question has many ramifications, we will focus here on a restricted family of extension operators. We begin with a justification for the restricted focus.

Definitions. It turns out that this problem arises in a number of instances which on the face of it appear quite different, but turn out to be unitarily equivalent. While we have in mind models for scattering of waves on a disconnected obstacle, and quantum mechanical transition probabilities, it will be convenient for us to select the version of problem (SAE) where $\mathcal{H} = L^2(\Omega)$ and Ω is a bounded open subset of the real line $\mathbb{R}$ with a finite number of components, i.e., Ω is a finite union of open disjoint intervals. We consider $\mathsf{D} := \frac{1}{2\pi i}\frac{d}{dx}$ corresponding to vanishing boundary conditions (the minimal operator). Then the deficiency indices are (n, n) when n is the number of components in Ω. In a different context, mathematical physics, the minimal operator was considered in [19].

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{H}}$. Let $\mathcal{D} \subset \mathcal{H}$ be a dense subspace in $\mathcal{H}$. A linear operator L defined on $\mathcal{D}$ is said to be *symmetric* (or *Hermitian*) iff

$$\langle Lf, g\rangle = \langle f, Lg\rangle \quad \text{for all } f, g \in \mathcal{D}.$$

In this case, the adjoint operator L^* is defined on a subspace domain(L^*) containing $\mathcal{D}$ and $L \subset L^*$, where "$\subset$" refers to containment of graphs.

If the dimensions of the two eigenspaces $\{f_\pm \in \text{domain}(L^*) : L^* f_\pm = \pm i f_\pm\}$ are equal (called *the deficiency indices*) then L has self-adjoint extensions. Every self-adjoint extension A of L must satisfy $L \subset A \subset L^*$ and any such A will be a restriction of L^*.

In Section 3, we offer a geometric model for the study of finite deficiency indices (n, n). While this work is directly related to recent work [23–25], our present focus is different, as are our themes. To make our present paper reasonably self-contained, it will be convenient for us to include here (Section 3) some basic lemmas needed in the proof of our main theorems. When n (the number of intervals in Ω) is fixed, the set of all self-adjoint extensions of D is in bijective correspondence with the group U_n of all