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Boundary regularity of suitable weak solution for the Navier–Stokes equations

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ABSTRACT

We study Hausdorff dimension of boundary singular set for suitable weak solutions of the incompressible Navier–Stokes equations. Like [2] for suitable weak solution in the interior, logarithmic improvement of Hausdorff dimension is proved. The pressure estimate of the inhomogeneous Stokes equations by Green potential due to Solonnikov [5] is important.

1. Introduction

In this paper, we prove boundary partial regularity of Navier–Stokes equations for suitable weak solution. After Scheffer [3] considered the boundary partial regularity of suitable weak solution, Seregin [4] established a criterion of ϵ regularity similar to [1] on the boundary.

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All the previous results rely on the boundary localized energy inequality and higher integrability of pressure such as $L^{3/2}$ of suitable weak solutions. From the definition of suitable weak solution, the pressure satisfies Poisson equation and is represented by Newtonian potential with density made of velocity. Therefore, we need complicated estimates of pressure in various forms. However, we obtain boundary localized estimates of velocity and pressure from the representation of solution by Green potential for the inhomogeneous Stokes equations due to Solonnikov [5].

Following [1] we introduce suitable weak solution. Although general boundary geometric conditions are important, we consider merely an initial boundary value problem in the half space $\Omega_T = (0,T) \times R^3_+$ of Navier–Stokes equations:

$$\frac{\partial}{\partial t}u - \nu\Delta u + \operatorname{div}(u \otimes u) + \nabla p = 0,$$
$$\operatorname{div} u = 0 \tag{1.1}$$

in $(0,T) \times R^3_+$ with an initial data

$$u(x,0) = u_0(x),$$

where the velocity fields u and u_0 are three dimension solenoidal vector fields and the pressure p is a scalar field. We let the viscosity $\nu = 1$ and meanwhile the terminal time T is not important in our argument. Due to viscosity, u satisfies no slip condition

$$u\big(x', 0, t\big) = 0 \quad \text{for } x' = \big(x^1, x^2\big) \in R^2,\ t > 0. \tag{1.2}$$

We define Lebesgue and Sobolev spaces by $L^p(\Omega) = \{f : \|f\|_{L^p(\Omega)} = (\int_\Omega |f|^p dx)^{1/p} < \infty\}$, $1 \le p < \infty$ and $H^1(\Omega) = \{f : \|f\|_{H^1(\Omega)} = (\int_\Omega |f|^2 + |\nabla f|^2 dx)^{1/2} < \infty\}$. $H^1_0(\Omega)$ is the H^1 closure of $C^\infty_0(\Omega)$ and $L^\infty(\Omega)$ is the set of essentially bounded measurable functions in Ω. We let $L^\infty(0,T : L^2(\Omega)) = \{f : ess \sup_{0<t<T} \|f\|_{L^2}(t) < \infty\}$ and $L^2(0,T : H^1(\Omega)) = \{f : \int_0^T \|f\|^2_{H^1}(t)dt < \infty\}$. Denote $z = (x,t)$ and $V(\Omega_T) = L^\infty(0,T : L^2(R^3_+)) \cap L^2(0,T : H^1_0(R^3_+))$. We say $(u,p) \in V \times L^{3/2}(\Omega_T)$ is suitable weak solution to the initial boundary value problem if for all $\phi \in C^\infty_0(R^3_+ \times R_+)$

$$\int u \cdot \phi_t dz + \int \nabla u : \nabla\phi dz + \int u \otimes u : \nabla\phi dz - \int p \operatorname{div}\phi dz = 0 \tag{1.3}$$

and u is weakly divergence free for almost all time, satisfies the localized energy inequality for almost all t

$$\int \big|u(x,t)\big|^2\phi dx + 2\int_0^t \int |\nabla u|^2 \phi dx ds$$
$$\le \int_0^t \int |u|^2(\phi_t + \Delta\phi) dx ds + \int_0^t \int \big(|u|^2 + 2p\big) u \cdot \phi dx ds \tag{1.4}$$