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Regularity of inverses of Sobolev deformations with finite surface energy

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A B S T R A C T

Let $\mathbf{u}$ be a Sobolev $W^{1,p}$ map from a bounded open set $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$. We assume $\mathbf{u}$ to satisfy some invertibility properties that are natural in the context of nonlinear elasticity, namely, the topological condition INV and the orientation-preserving constraint $\det D\mathbf{u} > 0$. These deformations may present cavitation, which is the phenomenon of void formation. We also assume that the surface created by the cavitation process has finite area. If $p > n - 1$, we show that a suitable defined inverse of $\mathbf{u}$ is a Sobolev map. A partial result is also given for the critical case $p = n - 1$. The proof relies on the techniques used in the study of cavitation.

1. Introduction

A classic question in analysis and topology is to find out the regularity of the inverse function $\mathbf{u}^{-1}$ in terms of the regularity of the original function $\mathbf{u}$. In particular, the issue of ascertaining the optimal Sobolev or BV regularity of $\mathbf{u}^{-1}$ given that of $\mathbf{u}$ has experienced a recent interest in the last decade. Most of the works in this question

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(see [18,19,24,20,8,17,27]) assume additionally that $\mathbf{u}$ is a homeomorphism. This implies, in particular, that $\mathbf{u}(\Omega)$ is open, so it makes sense to talk about a Sobolev or BV space over $\mathbf{u}(\Omega)$.

In the context of nonlinear elasticity, one assumes that $\mathbf{u}$ is in the Sobolev space $W^{1,p}$ for some $p > 1$, but the assumption that $\mathbf{u}$ is a homeomorphism is not acceptable in general. Indeed, while Ball [2] proved that if $p > n$ and if other integrability conditions hold then deformations are homeomorphisms, in the case when $p < n$ there are interesting deformations in $W^{1,p}$ that present singularities, and, in particular, are not continuous. One such type of singularity is that of *cavitation*, which is the process of formation of voids in solids (see [3]). In fact, determining the conditions on the stored-energy function under which cavitation occurs was an important part of the motivation for the papers [25,26,23,22] to study some regularity properties of a suitable defined inverse of $\mathbf{u}$; to be precise, the assumptions in [25,26,23] are incompatible with cavitation, while [22] does allow for cavitation. In those works, the deformation $\mathbf{u}$ was assumed to enjoy a certain property of invertibility much weaker than being a homeomorphism.

Following the steps of Müller and Spector [22], the authors [13–16] carried out an existence theory for deformations allowing for fracture and cavitation. As happened with [22] (and earlier with Šverák [25]), that analysis lent itself to a study of the inverse of $\mathbf{u}$. In particular, in [14] we proved an SBV regularity property of the inverse of an approximately differentiable map that was needed in order to carry out a geometric study of the surface created by the deformation. When the deformation $\mathbf{u}$ was assumed to be a Sobolev homeomorphism, it was shown in [15], as a by-product of the analysis of cavitation, that the inverse is actually Sobolev $W^{1,1}$. The same conclusion had been given by Csörnyei, Hencl and Malý [8], in fact, with weaker assumptions, using techniques of mappings of finite distortion.

In this paper we remove the assumption of being a homeomorphism; in particular, the deformations studied can present cavities. Specifically, we employ some techniques of [14,15] to show that, under some assumptions on $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ that are natural in the context of cavitation (namely, $\det D\mathbf{u} > 0$ a.e., the topological condition INV holds, $p \geq n-1$ and $\mathbf{u}$ has finite surface energy), an adequate definition $\tilde{\mathbf{u}}^{-1}$ of the inverse of $\mathbf{u}$ is a Sobolev map. A key ingredient is the use of the topological image $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ of $\mathbf{u}$ as the domain space for $\tilde{\mathbf{u}}^{-1}$. The topological image, which is defined as the set of points for which $\mathbf{u}$ has nonzero degree, coincides a.e. with the union of the image of $\mathbf{u}$ and the cavities created. The map $\tilde{\mathbf{u}}^{-1}$ is essentially the inverse of $\mathbf{u}$ outside the cavities, and it sends the whole cavity volume in the deformed configuration into the cavity point in the reference configuration. Thus, $\tilde{\mathbf{u}}^{-1}$ is not one-to-one a.e., but the amount of non-injectivity is well controlled.

If $p > n-1$, the set $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ is open and, in this case, we prove that $\tilde{\mathbf{u}}^{-1} \in W^{1,1}(\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega), \mathbb{R}^n)$. In the critical case $p = n-1$ the set $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ is not open in general. Nevertheless, we prove that the extension of $\tilde{\mathbf{u}}^{-1}$ by zero to $\mathbb{R}^n$ is an SBV function whose jump set does not intersect $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$; in particular, the restriction of the distributional derivative $D\tilde{\mathbf{u}}^{-1}$ to $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \Omega)$ is an L^1 function.