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Stein–Malliavin approximations for nonlinear functionals of random eigenfunctions on $\mathbb{S}^d$

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We investigate Stein–Malliavin approximations for nonlinear functionals of geometric interest for random eigenfunctions on the unit d-dimensional sphere $\mathbb{S}^d$, $d \geq 2$. All our results are established in the high energy limit, i.e. as the corresponding eigenvalues diverge. In particular, we prove a quantitative Central Limit Theorem for the excursion volume of Gaussian eigenfunctions; this goal is achieved by means of several results of independent interest, concerning the asymptotic analysis for the variance of moments of Gaussian eigenfunctions, the rates of convergence in various probability metrics for Hermite subordinated processes, and quantitative Central Limit Theorems for arbitrary polynomials of finite order or general, square-integrable, nonlinear transforms. Some related issues were already considered in the literature for the 2-dimensional case $\mathbb{S}^2$; our results are new or improve the existing bounds even in these special circumstances. Proofs are based on the asymptotic analysis of moments of all order for Gegenbauer polynomials, and make extensive use of the recent literature on so-called fourth-moment theorems by Nourdin and Peccati.

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1. Introduction

The characterization of the asymptotic behavior in the high energy limit, i.e. as the corresponding eigenvalues diverge, for geometric functionals of random eigenfunctions f on a compact manifold $\mathcal{M}$ is a topic which has recently drawn considerable attention. For instance, several papers have focused on the investigation of nodal sets, i.e. $f^{-1}(0)$, and nodal domains (connected components of the complement $\mathcal{M} \setminus f^{-1}(0)$) in Gaussian setups, see e.g. [35,13,14,7,6].

In particular, much effort has been devoted to the case of the 2-dimensional unit sphere $\mathbb{S}^2$: indeed, the asymptotic behavior of nodal lines and nodal domains for spherical Gaussian eigenfunctions (random spherical harmonics) has been studied in [26,37,38], whereas the area of excursion sets has been considered in [21,22]. Of course, boundary length and excursion area are just two instances of the so-called intrinsic volumes, or Lipschitz–Killing curvatures [1]. In the case of $\mathbb{S}^2$, the family of Lipschitz–Killing curvatures is completed by the Euler–Poincaré characteristic, which has been investigated in [10,11,20]. Most of these papers have considered the computation of asymptotic expected values and variances in the high energy limit; to the best of our knowledge, Central Limit Theorem (CLT) results have only been established for the so-called Defect (i.e. the difference of the measure of the positive and negative regions) in [22] and for the excursion area in [23].

This literature has been largely motivated by applications from Mathematical Physics. In particular, according to Berry's Universality conjecture [5], Gaussian monochromatic waves (similarly to e.g. random spherical harmonics) could model deterministic eigenfunctions on a "generic" manifold with or without boundary; this heuristic has strongly motivated the analysis of nodal sets of the former. On the other hand, it is also well-known that random eigenfunctions are the Fourier components of square integrable isotropic fields on manifolds. Spherical random fields are customarily used to model several data sets in astrophysics and cosmology; the analysis of polynomial transforms or geometric functionals of spherical random eigenfunctions has hence become a major statistical tool for these disciplines. For instance, analytic expectations on the values of geometric functionals can be used for testing the goodness of fit of theoretical models vs observational data (e.g., on Cosmic Microwave Background radiation, see [16,24] or the monograph [19]).

A CLT by itself can often provide little guidance to the actual distribution of random functionals, as it is only an asymptotic result with no information on the speed of convergence to the limiting distribution. More refined results indeed aim at the investigation of the asymptotic behavior for various probability metrics, such as Kolmogorov, Total Variation and Wasserstein distances (to be defined below). In this respect, a major development in the last few years has been provided by the so-called *fourth-moments literature*, which is summarized in the recent monograph [28]. In short, in this rapidly expanding literature it has been shown how to establish sharp bounds on probability distances between multiple stochastic integrals and the standard Gaussian distribution,