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Subspaces of C^∞ invariant under the differentiation ☆

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Let L be a proper differentiation invariant subspace of $C^\infty(a,b)$ such that the restriction operator $\frac{d}{dx}|_L$ has a discrete spectrum Λ (counting with multiplicities). We prove that L is spanned by functions vanishing outside some closed interval $I \subset (a,b)$ and monomial exponentials $x^k e^{\lambda x}$ corresponding to Λ if its density is strictly less than the critical value $\frac{|I|}{2\pi}$, and moreover, we show that the result is not necessarily true when the density of Λ equals the critical value. This answers a question posed by the first author and B. Korenblum. Finally, if the residual part of L is trivial, then L is spanned by the monomial exponentials it contains.

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1. Introduction

Consider the space $C^\infty(a,b)$ equipped with the usual topology of uniform convergence on compacta of each derivative $f^{(k)}$, $k = 0, 1, \dots$; more specifically, the topology given by any of the translation invariant metrics given below. Consider a sequence (I_j) of compact intervals with $\bigcup_j I_j = (a,b)$, denote by $\|\cdot\|_j$ the sup-norm over I_j and set

$$d(f,g) = \sum_{j,k=0}^{\infty} 2^{-j-k} \frac{\|f^{(k)} - g^{(k)}\|_j}{1 + \|f^{(k)} - g^{(k)}\|_j}.$$

The present paper concerns the structure of closed subspaces of $C^\infty(a,b)$ which are invariant for the differentiation operator $D = \frac{d}{dx}$. Our investigation follows the classical line, namely we are going to consider an appropriate version of *spectral synthesis* for these subspaces, which we now explain.

Strictly speaking, a continuous operator has the property of spectral synthesis if any non-trivial invariant subspace is generated by the root-vectors contained in it. The definition extends in an obvious way to families of commuting operators. The parade examples are the translation invariant subspaces of the locally convex space of continuous functions on the real line. These are now well understood due to the work of J. Delsarte [5], J.-P. Kahane [6] and L. Schwartz [10]. In the setting of entire functions translation-invariance is equivalent to complex differentiation invariance and the spectral synthesis property has been proved by L. Schwartz [11].

The structure of differentiation-invariant subspaces of $C^\infty(a,b)$ is more complicated and was only investigated recently in [1]. The reason for the additional complication is the presence of the following subspaces: Given a closed set $S \subset (a,b)$ let

$$L_S = \{f \in C^\infty(a,b) : \ f^{(k)}(S) = \{0\},\ k \geq 0\}. \tag{1.1}$$

In many cases these subspaces are nontrivial, and obviously, they contain no root-vector of D, since these functions are monomial exponentials, i.e. they have the form $x \to x^n e^{\lambda x}$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$. According to [1] the D-invariant subspaces of $C^\infty(a,b)$ can be classified in terms of the spectrum of the restriction of this operator. More precisely, given such a closed subspace L of $C^\infty(a,b)$ we have the following three alternatives:

(i) $\sigma(D|_L) = \mathbb{C}$,
(ii) $\sigma(D|_L) = \emptyset$,
(iii) $\sigma(D|_L)$ is a nonvoid discrete subset of $\mathbb{C}$ consisting of eigenvalues of D.

Very little is known about the structure of subspaces of the form (i). A concrete example is obtained by choosing the set S in (1.1) to consist of finitely (but at least two) many points, or disjoint intervals. The subspaces of type (ii) are called residual and