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Weak amenability for Fourier algebras of 1-connected nilpotent Lie groups

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A B S T R A C T

A special case of a conjecture raised by Forrest and Runde (2005) [10] asserts that the Fourier algebra of every non-abelian connected Lie group fails to be weakly amenable; this was already known to hold in the non-abelian compact cases, by earlier work of Johnson (1994) [13] and Plymen (unpublished note). In recent work (Choi and Ghandehari, 2014 [4]) the authors verified this conjecture for the real $ax+b$ group and hence, by structure theory, for any semisimple Lie group.

In this paper we verify the conjecture for all 1-connected, non-abelian nilpotent Lie groups, by reducing the problem to the case of the Heisenberg group. As in our previous paper, an explicit non-zero derivation is constructed on a dense subalgebra, and then shown to be bounded using harmonic analysis. *En route* we use the known fusion rules for Schrödinger representations to give a concrete realization of the "dual convolution" for this group as a kind of twisted, operator-valued convolution. We also give some partial results for solvable groups which give further evidence to support the general conjecture.

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1. Introduction

Fourier algebras of locally compact groups comprise an interesting class of Banach function algebras whose detailed structure remains somewhat mysterious, especially for groups which are neither compact nor abelian. It was observed by B.E. Forrest [9] that these algebras have no non-zero continuous point derivations. Nevertheless, as part of his seminal paper [13], B.E. Johnson constructed a continuous non-zero derivation from the Fourier algebra of SO(3) into a suitable Banach bimodule: in the language of [3], he proved that the Fourier algebra of SO(3) is not *weakly amenable*. This can be interpreted as evidence for some kind of weak form of differentiability or Hölder continuity for functions in the algebra.

Sufficient conditions for weak amenability were obtained in [10]: if G is locally compact and the connected component of its identity element is abelian, then its Fourier algebra $\mathrm{A}(G)$ is weakly amenable. Motivated by Johnson's result, the authors of [10] conjectured that this sufficient condition for weak amenability of $\mathrm{A}(G)$ is necessary. In particular, their conjecture implies that the Fourier algebra of any non-abelian connected Lie group is not weakly amenable; this was known at the time for compact Lie groups, but unknown for several natural examples including $\mathrm{SL}(2,\mathbb{R})$ and all the nilpotent cases.

This paper is a sequel to [4], which studied weak and cyclic amenability for Fourier algebras of certain connected Lie groups, and whose introduction contains further information on the history and context of the results mentioned above. In that paper we showed that the Fourier algebra of any connected, semisimple Lie group fails to be weakly amenable. The key to this result was to show that the Fourier algebra of the real $ax+b$ group is not weakly amenable, and this in turn was done by constructing an explicit non-zero derivation from the Fourier algebra to its dual. The derivation constructed in [4] is easily defined on a dense subalgebra, but showing that it extends continuously to the whole Fourier algebra required careful estimates provided by explicit orthogonality relations for certain coefficient functions of the real $ax+b$ group. We also proved, using similar techniques, that the Fourier algebra of the reduced Heisenberg group $\mathbb{H}_r$ is not weakly amenable. However, our methods were not able to handle the Fourier algebra of the "full" 3-dimensional Heisenberg group $\mathbb{H}$, which is a key example to consider when seeking to prove or refute the conjecture of Forrest and Runde.

In the present paper we develop techniques which allow us to fill this gap. The outcome is the following new result.

Theorem 1.1. *There exist a symmetric Banach bimodule* W *and a bounded, non-zero derivation* $D:\mathrm{A}(\mathbb{H})\to\mathsf{W}$. *Consequently,* $\mathrm{A}(\mathbb{H})$ *is not weakly amenable.*

This result then opens the way, via structure theory of Lie groups and Herz's restriction theorem for Fourier algebras, to the following more general statement.

Theorem 1.2. *Let* G *be a* 1*-connected Lie group. If* G *is also nilpotent and non-abelian, then* $\mathrm{A}(G)$ *is not weakly amenable.*