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Spectral measures with arbitrary Hausdorff dimensions ☆

Xin-Rong Dai [a], Qiyu Sun [b,*]

[a] School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou, 510275, China
[b] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

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ABSTRACT

In this paper, we consider spectral properties of Riesz product measures supported on homogeneous Cantor sets and we show the existence of spectral measures with arbitrary Hausdorff dimensions, including non-atomic zero-dimensional spectral measures and one-dimensional singular spectral measures.

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* Corresponding author.
E-mail addresses: daixr@mail.sysu.edu.cn (X.-R. Dai), qiyu.sun@ucf.edu (Q. Sun).

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1. Introduction

Given sequences $\mathcal{B} := \{b_n\}_{n=1}^{\infty}$ and $\mathcal{D} := \{d_n\}_{n=1}^{\infty}$ of positive integers that satisfy

$$1 < d_n < b_n, \quad n = 1, 2, \cdots, \tag{1.1}$$

we let

$$\rho_1 := 1 \quad \text{and} \quad \rho_n := \prod_{j=1}^{n-1} b_j \quad \text{for } n \geq 2, \tag{1.2}$$

and we define

$$C(\mathcal{B}, \mathcal{D}) := \sum_{n=1}^{\infty} \frac{\mathbb{Z}/d_n \cap [0, 1)}{\rho_n}. \tag{1.3}$$

The set $C(\mathcal{B}, \mathcal{D})$ is a *homogeneous Cantor set* contained in the interval $[0, \sum_{n=1}^{\infty}(d_n - 1)(d_n\rho_n)^{-1}]$. The reader may refer to [12,13,29] on homogeneous Cantor sets.

Define the Fourier transform $\hat{\mu}$ of a probability measure μ by $\hat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i\xi x} d\mu(x)$. In this paper, we consider the *Riesz product measure* $\mu_{\mathcal{B},\mathcal{D}}$ defined by

$$\widehat{\mu_{\mathcal{D},\mathcal{B}}}(\xi) := \prod_{n=1}^{\infty} H_{d_n}\left(\frac{\xi}{d_n\rho_n}\right), \tag{1.4}$$

where

$$H_m(\xi) := \frac{1}{m} \sum_{j=0}^{m-1} e^{-2\pi ij\xi} = \frac{1 - e^{-2\pi mi\xi}}{m(1 - e^{-2\pi i\xi})}, \quad m \geq 1.$$

The Riesz product measure $\mu_{\mathcal{B},\mathcal{D}}$ is supported on the homogeneous Cantor set $C(\mathcal{B}, \mathcal{D})$ [12,13], and it becomes the *Cantor measure* $\mu_{b,d}$ when $b_n = b$ and $d_n = d$ for all $n \geq 1$ [3–5,8].

A probability measure μ with compact support is said to be a *spectral measure* if there exists a countable set Λ of real numbers, called a *spectrum*, such that $\{e^{-2\pi i\lambda x} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. A classical example of spectral measures is the Lebesgue measure on $[0, 1]$, for which the set of integers is a spectrum. Spectral properties for a probability measure are one of fundamental problems in Fourier analysis and they have close connection to tiling as formulated in Fuglede's spectral set conjecture [14,17,18,20,22,23,31,32]. In 1998, Jorgensen and Pedersen [19] discovered the first families of non-atomic singular spectral measures, particularly Cantor measures $\mu_{b,2}$ with $4 \leq b \in 2\mathbb{Z}$. Since then, various singular spectral measures on self-similar/self-affine fractal sets have been found, see for instance [3–6,8,11,15–17,19,21,22,24,26,27,30,34]. In this paper, we consider spectral properties of Riesz product measures $\mu_{\mathcal{B},\mathcal{D}}$ supported on **non-self-similar** homogeneous Cantor sets $C(\mathcal{B}, \mathcal{D})$.