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Constructive solutions to Pólya–Schur problems

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A B S T R A C T

The general Pólya–Schur problem is to characterize linear operators on the space of univariate polynomials that preserve stability, where a polynomial is stable with respect to a region Ω in the complex plane if it has no zeros in Ω. Stable preserving operators have proven to be important in a variety of applications ranging from statistical mechanics to combinatorics, and variants of Pólya–Schur problems involving analytic functions are important in applications to signal processing. We present a structure theorem that bridges polynomial and analytic Pólya–Schur problems, providing constructive characterizations of stable-preserving operators for a general class of domains Ω. The structure theorem facilitates the solution of open Pólya–Schur problems in the classical setting, and provides constructive characterizations of stable preserving operators in cases where previously known characterizations are non-constructive. In the analytic setting, the structure theorem enables the explicit characterization of minimum-phase preserving operators on the half-line, a problem of importance in geophysical signal processing.

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1. Introduction

A univariate polynomial $p \in \mathbb{C}[z]$ is said to be stable with respect to a set $\Omega \subset \mathbb{C}$ if it has no zeros in Ω. Let $\mathcal{S}(\Omega)$ denote the set of all Ω-stable polynomials, together with 0. A linear mapping

$$A : \mathbb{C}[z] \to \mathbb{C}[z]$$

is stability-preserving with respect to Ω if $A(\mathcal{S}(\Omega)) \subset \mathcal{S}(\Omega)$; we denote the semigroup of all such mappings by $\mathscr{S}(\Omega)$.

The general Pólya–Schur problem is to characterize $\mathscr{S}(\Omega)$ for various $\Omega \subset \mathbb{C}$; the name harkens back to work by Pólya and Schur on multiplier sequences in the case $\Omega = \mathbb{C} \setminus \mathbb{R}$, [10]. Recently, Pólya–Schur problems have come into prominence for a variety of applications (see [12] for an overview), notably in the Ising model of statistical mechanics, where they underpin construction of Lee–Yang polynomials [2,11]. These results rely on characterizations of $\mathscr{S}(\Omega)$ for circular domains Ω and their boundaries presented in [3]. The latter work gives two types of characterizations: algebraic, in which a particular class of test functions—such as translates of monomials—determines whether a given operator preserves stability; and transcendental, whereby an operator preserves stability according to whether its characteristic function belongs to a certain analytic class.

Independently of the above developments, we have studied an analytic version of a Pólya–Schur problem in the context of the Hardy–Hilbert space $H^2 = H^2(\mathbb{D})$ on the unit disk, motivated by geophysical applications [5]. More precisely, the classical factorization theorem for H^2 expresses an arbitrary function as a product of an inner function, defined to have constant modulus 1 on the boundary circle, and an outer function, by definition a cyclic vector of the unilateral shift (see [9]). Thus a function $f \in H^2$ is outer if and only if the span of functions of the form $z^n f(z)$, where $n \geq 0$, is dense in H^2. Referring to functions of the form $z^n f(z)$ as shifted outer functions, [5] constructively characterizes continuous linear operators $A : H^2 \to H^2$ that preserve the class $\widetilde{\mathcal{O}} \subset H^2$ of all shifted outer functions. This is analogous to a Pólya–Schur problem because outer functions have no zeros in the open unit disk $\mathbb{D}$, and hence shifted outer functions have no zeros in the punctured disk $\mathbb{D} \setminus \{0\}$. The characterization of operators that preserve shifted outer functions translates via the inverse Z-transform to a characterization of operators acting on causal digital signals that preserve the class of delayed minimum-phase signals, a property of importance for seismic recordings (see [5] for full details).

The present paper stems from two principal objectives: (i) to bring the classical Pólya–Schur problems and their analytic analogues together within a common framework; and (ii) to extend the characterization of operators preserving delayed minimum-phase digital signals to operators acting on continuous signals in $L^2(\mathbb{R}_+)$, where the methods of [5] are insufficient. We prove a structure theorem that achieves both of these objectives and that furthermore solves an array of previously open Pólya–Schur problems. Let $\mathfrak{A}(\mathbb{D})$ denote the vector space of analytic functions on the open unit disk $\mathbb{D}$. Given a set $\Omega \subset \mathbb{D}$,