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Sets of multiplicity and closable multipliers on group algebras

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To the memory of William Arveson, with gratitude and admiration

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ABSTRACT

We undertake a detailed study of the sets of multiplicity in a second countable locally compact group G and their operator versions. We establish a symbolic calculus for normal completely bounded maps from the space $\mathcal{B}(L^2(G))$ of bounded linear operators on $L^2(G)$ into the von Neumann algebra $\mathrm{VN}(G)$ of G and use it to show that a closed subset $E \subseteq G$ is a set of multiplicity if and only if the set $E^* = \{(s,t) \in G \times G : ts^{-1} \in E\}$ is a set of operator multiplicity. Analogous results are established for M_1-sets and M_0-sets. We show that the property of being a set of multiplicity is preserved under various operations, including taking direct products, and establish an Inverse Image Theorem for such sets. We characterise the sets of finite width that are also sets of operator multiplicity, and show that every compact operator supported on a set of finite width can be approximated by sums of rank one operators supported on the same set. We show that, if G satisfies a mild approximation condition, pointwise multiplication by a given measurable function $\psi : G \to \mathbb{C}$ defines a closable multiplier on the reduced C^*-algebra $C_r^*(G)$ of G if and only if Schur multiplication by the function $N(\psi) : G \times G \to \mathbb{C}$, given by $N(\psi)(s,t) = \psi(ts^{-1})$, is a closable operator when

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viewed as a densely defined linear map on the space of compact operators on $L^2(G)$. Similar results are obtained for multipliers on VN(G).

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1. Introduction

The connections between Harmonic Analysis and the Theory of Operator Algebras have a long and illustrious history. With his pivotal paper [2], W.B. Arveson opened up a new avenue in that direction by introducing the notion of operator synthesis. The relation between operator synthesis and spectral synthesis for locally compact groups was explored in detail in [14,25,39,9,10], among others. In this connection, J. Froelich [14] studied the question of when the operator algebra associated with a commutative subspace lattice contains a non-zero compact operator. For any compact abelian group G and a closed subset $E \subseteq G$, he constructed a commutative subspace lattice $\mathcal{L}_E$, such that the corresponding operator algebra contains a non-zero compact operator if and only if E is a set of multiplicity in the sense of (commutative) Harmonic Analysis.

Recently, we observed in [34] a connection between sets of multiplicity and the closability of linear transformations that are a natural unbounded analogue of Schur multipliers. Motivated originally by Schur multiplication of matrices, Schur multipliers have played an important role in a number of contexts in Operator Theory, see *e.g.* [18] and [30]. In