Journal of Functional Analysis 268 (2015) 1509–1586

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

The infinite dimensional Evans function [☆]

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A R T I C L E   I N F O

Article history:
Received 31 October 2011
Accepted 23 November 2014
Available online 8 December 2014
Communicated by J. Coron

MSC:
primary 47B10, 47G10
secondary 34B27, 34L40

Keywords:
Fredholm determinants
Jost functions
Linear stability
Traveling waves

A B S T R A C T

We introduce generalized operator valued Jost solutions of first order ill-posed differential equations on Hilbert spaces. We then construct an infinite dimensional Evans function for abstract differential equations as a 2-modified Fredholm determinant of the operator obtained by adding the values at zero of the generalized operator valued Jost solutions. Next, we prove a formula that connects the 2-modified Evans determinant and the 2-modified determinant related to the Birman–Schwinger type operator associated to the ill-posed equation. Using this formula, we construct a holomorphic infinite dimensional Evans function for second order differential operators on infinite cylinders whose zeros are the eigenvalues of the differential operators.

☆ Based upon work supported by the US National Science Foundation under Grant Nos. DMS-0754705 and DMS-1067929, by the Research Board, and Research Council of the University of Missouri.
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http://dx.doi.org/10.1016/j.jfa.2014.11.020

1. Introduction

The finite dimensional Evans function is a widely used tool for detecting eigenvalues of the differential operators that appear after linearizing partial differential equations about such special solutions as steady states, traveling waves, etc. There is a big fascinating literature on the subject, we refer to [1,3,14,20,25,26,35,36] and more literature cited in review [30]. The construction of the *infinite dimensional* Evans function in a general setting is a longstanding open problem, although there is an extensive literature on this topic mainly related to problems on infinite cylinders, see [10,11,19,24] and the literature therein.

In this paper we introduce the Evans function for fairly general infinite dimensional systems using the (modified) Fredholm determinants of integral operators of the Birman–Schwinger type, continuing the line of analysis in [17] and [7,9,19,21,23]. In particular, we show that the Evans function is a holomorphic function of a spectral parameter, and that the zeros of the Evans function are the eigenvalues of the related differential operator.

To explain the main new ideas of the current work, we briefly review the line of arguments in [17]. The Evans function $E = E(\lambda)$ is associated in [17] to a pair of unperturbed and perturbed differential equations,

$$u'(t) = Au(t), \qquad u'(t) = \big(A + B(t)\big)u(t), \quad t \in \mathbb{R}. \tag{1.1}$$

Following [17], we assume for the moment that $A = A(\lambda)$ and $B(t)$ are $(d \times d)$ matrices, $u(t) \in \mathbb{C}^d$ and $\lambda \in \Omega \subset \mathbb{C}$ is a spectral parameter. Let us assume that $A = A(\lambda)$ has no spectrum on the imaginary axis, and denote by $\mathcal{Q}_+ = \mathcal{Q}_+(\lambda)$, respectively, $\mathcal{Q}_- = \mathcal{Q}_-(\lambda)$ the spectral projections of $A(\lambda)$ corresponding to the part of the spectrum $\sigma(A(\lambda))$ located in the left, respectively, in the right half-plane. Denoting the real parts of the eigenvalues of A by $\varkappa_j$ and numbering them so that $\cdots < \varkappa_{-2} < \varkappa_{-1} < 0 < \varkappa_1 < \varkappa_2 < \cdots$, and denoting by Q_j the spectral projection corresponding to the eigenvalues of A whose real parts are equal to $\varkappa_j$, we obtain finer decompositions $\mathcal{Q}_+ = \sum_{j<0} Q_j$ and $\mathcal{Q}_- = \sum_{j>0} Q_j$ invariant with respect to the semigroups $T_\pm(t) = e^{\pm tA_\pm}$, $t \geq 0$, where $A_\pm = \pm A_{|\operatorname{im} \mathcal{Q}_\pm}$. Since $\operatorname{Re} \sigma(A_{|\operatorname{im} \mathcal{Q}_+}) < 0$ and $\operatorname{Re} \sigma(-A_{|\operatorname{im} \mathcal{Q}_-}) < 0$, the operator $A = A_+ \oplus (-A_-)$ generates a stable bi-semigroup and $T_\pm(\cdot)Q_j$ are the matrix valued solutions of the unperturbed matrix equation $Y'(t) = AY(t)$, $t \in \mathbb{R}_\pm$, that decay to zero at the exponential rate $\varkappa_j$ as $t \to \pm\infty$ for $\pm j < 0$. The generalized matrix valued Jost solutions $Y_\pm^{(j)}(\cdot)$ are defined in [17] as the matrix solutions of the perturbed matrix equation $Y'(t) = (A + B(t))Y(t)$, $t \in \mathbb{R}_\pm$ that decay to zero exponentially as $t \to \pm\infty$ for $\pm j < 0$ so that the difference $Y_\pm^{(j)}(t) - T_\pm(t)Q_j$ decays to zero as $t \to \pm\infty$ faster than $e^{\varkappa_j t}$, $\pm j < 0$. As shown in [17] under natural assumptions on the perturbation, the generalized matrix valued Jost solutions exist, and then one can define the *Evans determinant*, $E = \det(\mathcal{Y}_+ + \mathcal{Y}_-)$, where $\mathcal{Y}_\pm = \sum_{\pm j<0} Y_\pm^{(j)}(0)$ are computed via the values of the generalized matrix valued Jost solutions at $t = 0$. The main result of [17] is an explicit formula,