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# Sum rules via large deviations

Fabrice Gamboa [a,*], Jan Nagel [b], Alain Rouault [c]

[a] *Université Paul Sabatier, Institut de Mathématiques de Toulouse, 118 route de Narbonne, 31062 Toulouse Cedex 9, France*

[b] *Technische Universität München, Fakultät für Mathematik, Boltzmannstr. 3, 85748 Garching, Germany*

[c] *Université Versailles-Saint-Quentin, LMV UMR 8100, 45 Avenue des Etats-Unis, 78035-Versailles Cedex, France*

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In the theory of orthogonal polynomials, sum rules are remarkable relationships between a functional defined on a subset of all probability measures involving the reverse Kullback–Leibler divergence with respect to a particular distribution and recursion coefficients related to the orthogonal polynomial construction. Killip and Simon [24] have given a revival interest to this subject by showing a quite surprising sum rule for measures dominating the semicircular distribution on $[-2, 2]$. This sum rule includes a contribution of the atomic part of the measure away from $[-2, 2]$. In this paper, we recover this sum rule by using probabilistic tools on random matrices. Furthermore, we obtain new (up to our knowledge) *magic* sum rules for the reverse Kullback–Leibler divergence with respect to the Marchenko–Pastur or Kesten–McKay distributions. As in the semicircular case, these formulas include a contribution of the atomic part appearing away from the support of the reference measure.



* Corresponding author.
*E-mail addresses:* gamboa@math.univ-toulouse.fr (F. Gamboa), jan.nagel@tum.de (J. Nagel), alain.rouault@uvsq.fr (A. Rouault).

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## 1. Introduction

### 1.1. Szegő–Verblunsky theorem and sum rules

A very famous result in the theory of orthogonal polynomial on the unit circle (OPUC) is the Szegő–Verblunsky theorem (see [32] Theorem 1.8.6 p. 29). It concerns a deep relationship between the coefficients involved in the construction of the orthogonal polynomial sequence of a measure supported by the unit circle and its logarithmic entropy. More precisely, the inductive relation between two successive monic orthogonal polynomials $\phi_{n+1}$ and $\phi_n$ ($\deg \phi_n = n$, $n \geq 0$) associated with a probability measure $\mu$ on the unit circle $\mathbb{T}$ supported by at least $n+1$ points involves a complex number $\alpha_n$ and may be written as

$$\phi_{n+1}(z) = z\phi_n(z) - \overline{\alpha}_n \phi_n^*(z) \text{ where } \phi_n^*(z) := z^n \overline{\phi_n(1/\bar{z})}. \tag{1.1}$$

The complex number $\alpha_n = -\overline{\phi_{n+1}(0)}$ is the so-called Verblunsky coefficient. In other contexts, it is also called Schur, Levinson, Szegő coefficient or even canonical moment [13].

The Szegő–Verblunsky theorem is the identity

$$\frac{1}{2\pi} \int_0^{2\pi} \log g_\mu(\theta) d\theta = \sum_{n \geq 0} \log(1 - |\alpha_n|^2), \tag{1.2}$$

where the Lebesgue decomposition of $\mu$ with respect to the uniform measure $d\theta/2\pi$ on $\mathbb{T}$ is

$$d\mu(\theta) = g_\mu(\theta) \frac{d\theta}{2\pi} + d\mu_s(\theta),$$

and where both sides of (1.2) are simultaneously finite or infinite. An exhaustive overview and the genesis tale of this crucial theorem may be found in the very nice book of Simon [32]. The identity (1.2) is one of the most representative example of a sum rule (or trace formula): it connects the coefficients of an operator [22] to its spectral data. There are various analytical methods of proof (see Chapter 1 in [32]) and a probabilistic one (see Section 5.2 of [18]).

In the theory of orthogonal polynomials on the real line (OPRL), given a probability measure $\mu$ with an infinite support, a.k.a. nontrivial case (resp. with a finite support consisting of $n$ points, a.k.a. trivial case), the orthonormal polynomials (with positive leading coefficients) obtained by applying the orthonormalizing Gram–Schmidt procedure to the sequence $1, x, x^2, \dots$ obey the recursion relation

$$xp_k(x) = a_{k+1}p_{k+1}(x) + b_{k+1}p_k(x) + a_k p_{k-1}(x) \tag{1.3}$$

for $k \geq 0$ (resp. for $0 \leq k \leq n-1$) where the Jacobi parameters satisfy $b_k \in \mathbb{R}, a_k > 0$. Notice that here the orthogonal polynomials are not monic but normalized in $L^2(\mu)$.