Journal of Functional Analysis 270 (2016) 621–648

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Functions operating on modulation spaces and nonlinear dispersive equations

Divyang G. Bhimani, P.K. Ratnakumar *

Harish-Chandra Research Institute, Allahabad 211019, India

ARTICLE INFO

Article history:
Received 3 March 2015
Accepted 29 October 2015
Available online 16 November 2015
Communicated by I. Rodnianski

MSC:
primary: 2B35
secondary: 42B37, 47H30, 35A01

Keywords:
Nonlinear operation
Modulation spaces
Dispersive equation
Local well-posedness

ABSTRACT

The aim of this paper is two fold. We show that if a complex function F on $\mathbb{C}$ operates in the modulation spaces $M^{p,1}(\mathbb{R}^n)$ by composition, then F is real analytic on $\mathbb{R}^2 \approx \mathbb{C}$. This answers negatively, the open question posed in Ruzhansky et al. (2012) [21], regarding the general power type nonlinearity of the form $|u|^\alpha u$. We also characterise the functions that operate in the modulation space $M^{1,1}(\mathbb{R}^n)$.
The local well-posedness of the NLS, NLW and NLKG equations for the 'real entire' nonlinearities are also studied in some weighted modulation spaces $M^{p,q}_s(\mathbb{R}^n)$.

1. Introduction

A classical theorem of Wiener [35] and Lévy [18] tells us that if $f \in A(\mathbb{T})$, where $A(\mathbb{T})$ denotes the class of all functions on the unit circle whose Fourier series is absolutely convergent, and F is defined and analytic on the range of f, then $F(f) \in A(\mathbb{T})$; where $F(f)$ is the composition of functions F and f. Recently in this direction, M. Sugimoto et al. have shown in [26], that the Wiener–Lévy type theorem is valid in the realm of

* Corresponding author. Fax: +91 532 2567 748.
E-mail addresses: divyang@hri.res.in (D.G. Bhimani), ratnapk@hri.res.in (P.K. Ratnakumar).

http://dx.doi.org/10.1016/j.jfa.2015.10.017
0022-1236/

weighted modulation spaces $M_s^{p,q}(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$, $s > n/q'$, $1/q + 1/q' = 1$, for complex entire function F vanishing at 0. See Section 2, for the definition of modulation spaces.

For $F : \mathbb{R}^2 \to \mathbb{C}$, we denote by T_F the operator $T_F(f) := F(f)$. One of the results that we prove in this paper, namely Theorem 3.9, says that if F is real entire and $F(0) = 0$, then T_F maps X to itself, that is, $T_F(X) \subset X$; where $X = M^{p,1}(\mathbb{R}^n)$, $(1 \leq p \leq \infty)$ or $X = M_s^{p,q}(\mathbb{R}^n)$, $(1 \leq p, q \leq \infty, s > n/q')$; here the proof relies on the multiplication algebra property of X. In fact, we have gone a bit further, and shown that under the weaker hypothesis on F, namely that F is real analytic on $\mathbb{R}^2$ and $F(0) = 0$, then $T_F\left(M^{1,1}(\mathbb{R}^n)\right) \subset M^{1,1}(\mathbb{R}^n)$; the proof relies on the invariant property of the modulation space $M^{1,1}(\mathbb{R}^n)$ under the Fourier transform. This invariance is not available for $M^{p,1}(\mathbb{R}^n)$, when $p > 1$.

Now the natural question is: what are all functions F such that T_F takes $M^{p,1}(\mathbb{R}^n)$ to itself? In Theorem 3.4, we show that T_F maps $M^{p,1}(\mathbb{R}^n)$ to itself, then F must be real analytic on $\mathbb{R}^2$. The proof relies on the "localized" version of "time–frequency" spaces, which can be identified with the Fourier algebra on the torus $A(\mathbb{T}^n)$. As a consequence of the above two results, we obtain a characterisation theorem for the functions F that operates in $M^{1,1}(\mathbb{R}^n)$. In fact, we show that the real analytic functions are the only ones with the desired mapping property on $M^{1,1}(\mathbb{R}^n)$, see Theorem 1.3.

Our motivation for studying the above problem started with analysing well-posedness results for the nonlinear Schrödinger equation (*NLS*) for power type nonlinearities in Lebesgue spaces $L^p(\mathbb{R}^n)$. Consider the initial value problem

$$(NLS) \quad i\frac{\partial}{\partial t}u(x,t) + \triangle_x u(x,t) = F(u(x,t)), \quad u(x,0) = u_0(x),$$

where $\triangle_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $i = \sqrt{-1}$, u_0 is a complex valued function on $\mathbb{R}^n$ and the nonlinearity is given by a complex function F on $\mathbb{C}$.

The theory of *NLS* and other general nonlinear dispersive equations (4.2), (4.3) (local and global existence) is vast and has been studied extensively by many authors, see for instance [10]. Almost exclusively, well-posedness has been established in energy space $H^1(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$. The techniques have been restricted to L^2 Sobolev spaces because of the crucial role played by the Fourier transform in the analysis of partial differential equations.

The modern approach to study the well-posedness of dispersive equations, is through the Strichartz estimates [24], satisfied by the linear propagator for the associated dispersive equation. We refer to the far reaching generalisation of Strichartz estimates in [15] by M. Keel and T. Tao, and also [27,4] and the references therein.

However, much less is known for the same problem with L^p initial data, for $p \neq 2$. In fact, for any $t \neq 0$, $e^{it\Delta}$ is unbounded in any L^p, $p \neq 2$, which makes it difficult even to solve the free linear Schrödinger equation in L^p if $p \neq 2$. Perhaps the first attempt in this direction started in the work of Vargas and Vega [30], where they