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Discrete taut strings and real interpolation

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ABSTRACT

Classical taut strings and their multidimensional generalizations appear in a broad range of applications. In this paper we suggest a general approach based on the K-functional of real interpolation that provides a unifying framework of existing theories and extend the range of applications of taut strings. More exactly, we introduce the notion of invariant K-minimal sets, explain their connection to taut strings and characterize all bounded, closed and convex sets in $\mathbb{R}^n$ that are invariant K-minimal with respect to the couple (ℓ^1, ℓ^∞).

0. Introduction

Let $a = x_0 < x_1 < \ldots < x_n = b$ be $n+1$ points on the real line, $n \geq 1$, given by

$$x_i = a + \frac{i(b-a)}{n}, i = 0, 1, \ldots, n.$$

Let us consider two continuous functions $F \leq G$ on the interval $[a, b]$ that are linear on the intervals $[a, x_1]$, $[x_1, x_2]$,..., $[x_{n-1}, b]$. We suppose that

$$F(a) = G(a), F(b) = G(b).$$

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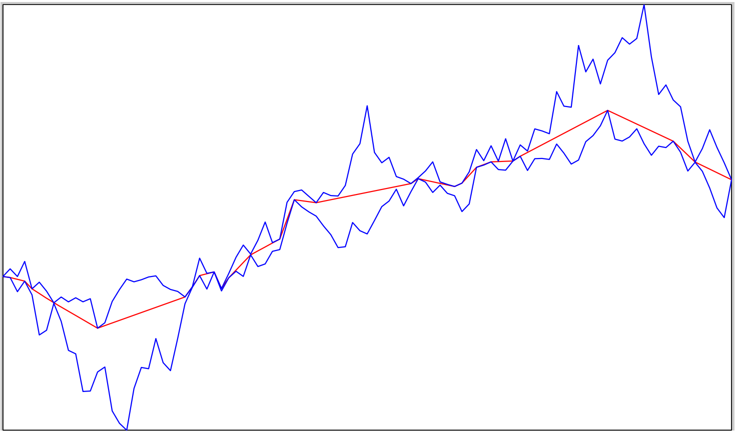

Fig. 1. Taut string (red) inside the corridor $\Gamma_{F,G}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Let $\Gamma_{F,G}$ denote the set of all continuous piecewise linear functions f on $[a, b]$ with nodes in x_i, $i = 0, 1, \dots, n$, and which satisfy the inequalities $F \le f \le G$. A function $f_* \in \Gamma_{F,G}$ is called taut string if it has minimal length among all functions $f \in \Gamma_{F,G}$, i.e.

$$\int_a^b \sqrt{1 + (f_*'(x))^2}\,dx = \inf_{f \in \Gamma_{F,G}} \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

For an illustration of the taut string, see Fig. 1.

The notion of taut string was introduced by G.B. Dantzig in 1971, see [3]. Dantzig notes that he first presented taut strings in R. Bellman's seminar at RAND Corporation in 1952 in connection with problems in optimal control. Later on taut strings and their one- and multidimensional generalizations have been used in different applied problems, in particular in statistics, see e.g. [1] and [12], and image processing, see [15]. Recently, new applications to stochastic processes, see [11], and communication theory, see [17] and [16], have been found.

Our interest in taut strings is based on the following connection to the theory of real interpolation. Consider the set

$$\Omega := \left\{ u = (u_i) \in \mathbb{R}^n : u_i = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}},\ i = 1, \dots, n, f \in \Gamma_{F,G} \right\}. \qquad (1)$$

It can be shown, see Theorem 5.1 below, that the element $u_* \in \Omega$ with $u_{*,i} = \frac{f_*(x_i) - f_*(x_{i-1})}{x_i - x_{i-1}}$, i.e. the vector with elements corresponding to the values of the piecewise constant derivative f_*' of the taut string $f_* \in \Gamma_{F,G}$, has minimal K-functional in Ω with respect to the couple (ℓ^1, ℓ^∞) on $\mathbb{R}^n$ (everywhere below the spaces ℓ^p are considered on $\mathbb{R}^n$). Hence, for *all* exact interpolation spaces of (ℓ^1, ℓ^∞) generated by the K-method,