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Non-compactness and infinite number of conformal initial data sets in high dimensions

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ABSTRACT

On any closed Riemannian manifold of dimension greater than 7, we construct examples of background physical coefficients for which the Einstein–Lichnerowicz equation possesses a non-compact set of positive solutions. This yields in particular the existence of an infinite number of positive solutions in such cases.

1. Introduction

Let (M, g) be a closed compact Riemannian manifold of dimension $n \geq 6$. We investigate in this work non-compactness issues for the set of positive solutions of the Einstein–Lichnerowicz equation in M, which writes as follows:

$$\triangle_g u + hu = f u^{2^*-1} + \frac{a}{u^{2^*+1}}. \tag{1.1}$$

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Here h, f, a are given functions in M such that $\triangle_g + h$ is coercive, $f > 0$ and $a \geq 0$, $a \not\equiv 0$ and $2^* = \frac{2n}{n-2}$ is the critical exponent for the embedding of the Sobolev space $H^1(M)$ into Lebesgue spaces. Equation (1.1) arises in the initial-value problem in General Relativity, when one looks for initial-data sets for the Einstein equations *via* the conformal method. The determination of constant-mean-curvature initial data sets amounts to the resolution of equation (1.1) in the prominent case where the coefficients h, f, a take the following form:

$$h = c_n \left(S_g - |\nabla \psi|_g^2 \right) , \quad f = c_n \left(2V(\psi) - \frac{n-1}{n} \tau^2 \right) , \quad a = \pi^2, \tag{1.2}$$

where $c_n = \frac{n-2}{4(n-1)}$. In (1.2), $V : \mathbb{R} \to \mathbb{R}$ is a potential, $\psi : M \to \mathbb{R}$ is a scalar-field, $\tau \in \mathbb{R}$ is the mean curvature and S_g is the scalar curvature of g. Physically speaking, ψ and π represent respectively the restriction of the ambient scalar-field and of its time-derivative to M and τ is the mean curvature of the Cauchy hypersurface M embedded in the space-time. See Bartnik–Isenberg [2] for a survey reference on the constraint equations.

Following the terminology introduced in Premoselli [23], in this work we consider equation (1.1) in the so-called *focusing case*, defined as:

$$\text{focusing case: } f > 0 \text{ in } M. \tag{1.3}$$

Since $a \geq 0$, standard variational arguments show that the coercivity of $\triangle_g + h$ is a necessary condition for (1.1) to possess smooth positive solutions. The existence of solutions of (1.1) in the focusing case was first obtained in Hebey–Pacard–Pollack [14] and multiplicity issues were later on investigated in Ma–Wei [19], Premoselli [25] and Holst–Meier [16]. Existence results for the non-constant-mean-curvature generalization of (1.1) in the physical case (1.2) are in Premoselli [24].

Stability issues for equation (1.1) in the focusing case have been investigated in Druet–Hebey [10], Hebey–Veronelli [15] and Premoselli [25]. The stability of the general conformal constraint system in all dimensions has been investigated by Druet–Premoselli [11] and Premoselli [23]. In the specific case of equation (1.1), the results of Druet–Hebey [10] and Premoselli [23] yield in particular that in dimensions $n \geq 6$, equation (1.1) is stable with respect to perturbations of its coefficients as soon as there holds in M:

$$h - c_n S_g + \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_g f}{f} < 0, \tag{1.4}$$

where S_g denotes the scalar curvature of g. More precisely, if (1.4) holds, then for any sequences $(h_\alpha)_\alpha$, $(f_\alpha)_\alpha$ and $(a_\alpha)_\alpha$, with $a_\alpha \geq 0$ satisfying:

$$\|h_\alpha - h\|_{C^0(M)} + \|f_\alpha - f\|_{C^2(M)} + \|a_\alpha - a\|_{C^0(M)} \to 0$$