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Sharp comparison and maximum principles via horizontal normal mapping in the Heisenberg group

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A B S T R A C T

In this paper we solve a problem raised by Gutiérrez and Montanari about comparison principles for H-convex functions on subdomains of Heisenberg groups. Our approach is based on the notion of the sub-Riemannian horizontal normal mapping and uses degree theory for set-valued maps. The statement of the comparison principle combined with a Harnack inequality is applied to prove the Aleksandrov-type maximum principle, describing the correct boundary behavior of continuous H-convex functions vanishing at the boundary of horizontally bounded subdomains of Heisenberg groups. This result answers a question by Garofalo and Tournier. The sharpness of our results are illustrated by examples.

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1. Introduction

1.1. Motivation

It is well known that convex functions defined on subdomains of $\mathbb{R}^n$ are locally Lipschitz continuous and almost everywhere twice differentiable. Moreover, the celebrated maximum principle due to Aleksandrov provides a global regularity result for convex functions that are continuous on the closure and are vanishing on the boundary of the domain. More precisely, if $\Omega \subset \mathbb{R}^n$ is a bounded open and convex domain, and $u \in C(\overline{\Omega})$ is convex with $u = 0$ on $\partial\Omega$, then

$$|u(\xi_0)|^n \le C_n \mathrm{dist}(\xi_0, \partial\Omega)\mathrm{diam}(\Omega)^{n-1}\mathcal{L}^n(\partial u(\Omega)),\ \forall \xi_0 \in \Omega, \tag{1.1}$$

where $C_n > 0$ is a constant depending only on the dimension n. In the above expression the notation $\mathcal{L}^n(\partial u(\Omega))$ stands for the measure of the range of the so-called normal mapping of u. To define this concept we need first the subdifferential $\partial u(\xi_0)$ of u at ξ_0, given by

$$\partial u(\xi_0) = \{p \in \mathbb{R}^n : u(\xi) \ge u(\xi_0) + p \cdot (\xi - \xi_0),\ \forall \xi \in \Omega\},$$

where '·' is the usual inner product in $\mathbb{R}^n$. The range of the normal mapping of u is defined by

$$\partial u(\Omega) = \bigcup_{\xi \in \Omega} \partial u(\xi).$$