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# Sharp comparison and maximum principles via horizontal normal mapping in the Heisenberg group



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## ABSTRACT

In this paper we solve a problem raised by Gutiérrez and Montanari about comparison principles for  $H$ -convex functions on subdomains of Heisenberg groups. Our approach is based on the notion of the sub-Riemannian horizontal normal mapping and uses degree theory for set-valued maps. The statement of the comparison principle combined with a Harnack inequality is applied to prove the Aleksandrov-type maximum principle, describing the correct boundary behavior of continuous  $H$ -convex functions vanishing at the boundary of horizontally bounded subdomains of Heisenberg groups. This result answers a question by Garofalo and Tournier. The sharpness of our results are illustrated by examples.

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**1. Introduction**

*1.1. Motivation*

It is well known that convex functions defined on subdomains of  $\mathbb{R}^n$  are locally Lipschitz continuous and almost everywhere twice differentiable. Moreover, the celebrated maximum principle due to Aleksandrov provides a global regularity result for convex functions that are continuous on the closure and are vanishing on the boundary of the domain. More precisely, if  $\Omega \subset \mathbb{R}^n$  is a bounded open and convex domain, and  $u \in C(\bar{\Omega})$  is convex with  $u = 0$  on  $\partial\Omega$ , then

$$|u(\xi_0)|^n \leq C_n \text{dist}(\xi_0, \partial\Omega) \text{diam}(\Omega)^{n-1} \mathcal{L}^n(\partial u(\Omega)), \quad \forall \xi_0 \in \Omega, \tag{1.1}$$

where  $C_n > 0$  is a constant depending only on the dimension  $n$ . In the above expression the notation  $\mathcal{L}^n(\partial u(\Omega))$  stands for the measure of the range of the so-called normal mapping of  $u$ . To define this concept we need first the subdifferential  $\partial u(\xi_0)$  of  $u$  at  $\xi_0$ , given by

$$\partial u(\xi_0) = \{p \in \mathbb{R}^n : u(\xi) \geq u(\xi_0) + p \cdot (\xi - \xi_0), \quad \forall \xi \in \Omega\},$$

where ‘ $\cdot$ ’ is the usual inner product in  $\mathbb{R}^n$ . The range of the normal mapping of  $u$  is defined by

$$\partial u(\Omega) = \bigcup_{\xi \in \Omega} \partial u(\xi).$$

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