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Implicit/inverse function theorems for free noncommutative functions ☆

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We prove an implicit function theorem and an inverse function theorem for free noncommutative functions over operator spaces and on the set of nilpotent matrices. We apply these results to study dependence of the solution of the initial value problem for ODEs in noncommutative spaces on the initial data and to extremal problems with noncommutative constraints.

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1. Introduction and statements of the results

1.1. Free NC functions

A free noncommutative (nc) function is a mapping defined on the set of matrices of all sizes which respects direct sums and similarities, or equivalently, respects intertwinings. Examples include but are not limited to nc polynomials, power series, and matrix-valued rational expressions. The theory of free nc functions was first introduced in the articles of Joseph L. Taylor [23,24]. It was further developed by D.-V. Voiculescu [25,26] for the needs of free probability. Various aspects of nc functions[1] have been studied by Helton [7], Helton, Klep, and McCullough [8,9], Helton and McCullough [10], Helton and Putinar [11], Popescu [20,21], Muhly and Solel [16], Agler and McCarthy [2,3], Agler and Young [4], the second author and Vinnikov [12,14], and others.

In the book of the second author and Victor Vinnikov [13], the theory has been put on a systematic foundation. The nc difference–differential calculus has been developed for studying various questions of nc analysis; in particular, the classical (commutative) theory of analytic functions was extended to a nc setting. It has been established that very mild assumptions of local boundedness of nc functions imply analyticity.

We provide the reader with some basic definitions from [13]. Let $\mathcal{R}$ be a unital commutative ring. For a module $\mathcal{M}$ over $\mathcal{R}$, we define the *nc space over* $\mathcal{M}$,

$$\mathcal{M}_{\rm nc} := \coprod_{n=1}^{\infty} \mathcal{M}^{n\times n}. \tag{1.1}$$

A subset $\Omega \subseteq \mathcal{M}_{\rm nc}$ is called a *nc set* if it is closed under direct sums; that is, denoting $\Omega_n = \Omega \cap \mathcal{M}^{n\times n}$, we have

$$X \in \Omega_n, Y \in \Omega_m \Longrightarrow X \oplus Y := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}. \tag{1.2}$$

Matrices over $\mathcal{R}$ act from the right and from the left on matrices over $\mathcal{M}$ by the standard rules of matrix multiplication: if $T \in \mathcal{R}^{r\times p}$ and $S \in \mathcal{R}^{p\times s}$, then for $X \in \mathcal{M}^{p\times p}$ we have

$$TX \in \mathcal{M}^{r\times p}, \quad XS \in \mathcal{M}^{p\times s}.$$

In the special case where $\mathcal{M} = \mathcal{R}^d$, we identify matrices over $\mathcal{M}$ with d-tuples of matrices over $\mathcal{R}$:

$$(\mathcal{R}^d)^{p\times q} \cong (\mathcal{R}^{p\times q})^d.$$

[1] Here and in the rest of the paper we will omit the word “free” for short.