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Cuntz–Pimsner algebras, crossed products, and K-theory

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ABSTRACT

Suppose A is a C^*-algebra and H is a C^*-correspondence over A. If H is regular in the sense that the left action of A is faithful and is given by compact operators, then we compute the K-theory of $\mathcal{O}_A(H) \rtimes \mathbb{T}$ where the action is the usual gauge action. The case where A is an AF-algebra is carefully analyzed. In particular, if A is AF, we show $\mathcal{O}_A(H) \rtimes \mathbb{T}$ is AF. Combining this with Takai duality and an AF-embedding theorem of N. Brown, we show the conditions AF-embeddability, quasidiagonality, and stable finiteness are equivalent for $\mathcal{O}_A(H)$. If H is also assumed to be regular, these finiteness conditions can be characterized in terms of the ordered K-theory of A.

1. Introduction

A C^*-correspondence consists of C^*-algebras A and B and an A–B bimodule H together with a complete B-valued inner product which satisfies certain conditions. In this case, H can be thought of as a multi-valued, partially defined morphism from A to B. In fact if H is "row-finite" then H induces a positive group homomorphism

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$[H] : K_0(A) \to K_0(B)$ (see Section 4). In the special case when $A = B$, Pimsner introduced in [14] a certain C^*-algebra $\mathcal{O}_A(H)$, now called the Cuntz–Pimsner algebra, which can be thought of as the crossed product of A by the generalized morphism H (see also [7,8] for example).

The class of Cuntz–Pimsner algebras includes many naturally occurring C^*-algebras such as crossed products by $\mathbb{Z}$, crossed products by partial automorphisms, graph algebras, and much more (see [7, Section 3] for other examples). Much of the structure of $\mathcal{O}_A(H)$ can be recovered from the underlying correspondence H and the algebra A. For instance, the ideal structure of $\mathcal{O}_A(H)$ is extensively studied in [10], and there is a Pimsner–Voiculescu six-term exact sequence in K-theory for $\mathcal{O}_A(H)$ given in [8, Theorem 8.6] (originally shown in [14, Theorem 4.9] with mild hypotheses).

There is a natural gauge action γ of $\mathbb{T}$ on $\mathcal{O}_A(H)$. By Takai duality, $\mathcal{O}_A(H)$ is Morita equivalent to $(\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}) \rtimes_{\hat{\gamma}} \mathbb{Z}$. Thus understanding the structure of $\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}$ can give some insight into the structure of the algebra $\mathcal{O}_A(H)$. In this note, we calculate the K-theory of $\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}$. In particular, we have the following result (see Section 5 for a proof).

Theorem A. *Suppose A is a C^*-algebra and H is a row-finite, faithful C^*-correspondence over A. Then*

$$K_*(\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}) \cong \varinjlim (K_*(A), [H]).$$

Moreover, the isomorphism preserves the order structure on K_0 and intertwines the automorphism $[\hat{\gamma}]$ on $K_*(\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T})$ and the automorphism of $\varinjlim (K_*(A), [H])$ induced by $[H]$.

The proof involves a certain skew product construction similar to the graph skew product construction developed by Kumjian and Pask in [11] for graph C^*-algebras. In particular, we build correspondences H^n over C^*-algebras A^n for $n \in \mathbb{Z} \cup \{\infty\}$ such that

$$\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T} \cong \mathcal{O}_{A^\infty}(H^\infty) \cong \varinjlim \mathcal{O}_{A^n}(H^n).$$

The K-theory of $\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}$ is calculated by showing that each $\mathcal{O}_{A^n}(H^n)$, is Morita equivalent to A. The result then follows by the continuity and stability of K-theory.

The case where A is an AF-algebra is also examined. In particular, we have the following result (see Section 6).

Theorem B. *If A is an AF-algebra and H is a separable C^*-correspondence over A, then $\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}$ is AF.*

We end this note with some applications to AF-embeddability. In [1], N. Brown characterized the AF-embeddability of crossed products of AF-algebras by the integers (see