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# Laplace operators on the cone of Radon measures

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A B S T R A C T

We consider the infinite-dimensional Lie group $\mathfrak{G}$ which is the semidirect product of the group of compactly supported diffeomorphisms of a Riemannian manifold $X$ and the commutative multiplicative group of functions on $X$. The group $\mathfrak{G}$ naturally acts on the space $\mathbb{M}(X)$ of Radon measures on $X$. We would like to define a Laplace operator associated with a natural representation of $\mathfrak{G}$ in $L^2(\mathbb{M}(X), \mu)$. Here $\mu$ is assumed to be the law of a measure-valued Lévy process. A unitary representation of the group cannot be determined, since the measure $\mu$ is not quasi-invariant with respect to the action of the group $\mathfrak{G}$. Consequently, operators of a representation of the Lie algebra and its universal enveloping algebra (in particular, a Laplace operator) are not defined. Nevertheless, we determine the Laplace operator by using a special property of the action of the group $\mathfrak{G}$ (a partial quasi-invariance). We further prove the essential self-adjointness of the Laplace

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operator. Finally, we explicitly construct a diffusion process on $\mathbb{M}(X)$ whose generator is the Laplace operator.


## 1. Introduction

In the representation theory, so-called quasi-regular representations of a group $\mathfrak{G}$ in a space $L^2(\Omega, \mu)$ play an important role. Here $\Omega$ is a homogeneous space for the group $\mathfrak{G}$ and $\mu$ is a (probability) measure that is quasi-invariant with respect to the action of $\mathfrak{G}$ on $\Omega$. However, in the case studied in this paper as well as in similar cases, the measure is not quasi-invariant with respect to the action of the group, so that one cannot define a quasi-regular unitary representation of the group. Hence, the problem of construction of a representation of the Lie algebra, of a Laplace operator and other operators from the universal enveloping algebra is highly non-trivial. Moreover, we will deal with the situation in which the representation of the Lie algebra cannot be realized but, nevertheless, the Laplace operator may be defined correctly.

An important example of a quasi-regular representation is the following case. Let $X$ be a smooth, noncompact Riemannian manifold, and let $\mathfrak{G} = \mathrm{Diff}_0(X)$, the group of $C^\infty$ diffeomorphisms of $X$ that are equal to the identity outside a compact set. Let $\Omega$ be the space $\Gamma(X)$ of locally finite subsets (configurations) in $X$, and let $\mu$ be the Poisson measure on $\Gamma(X)$. Then the Poisson measure is quasi-invariant with respect to the action of $\mathrm{Diff}_0(X)$ on $\Gamma(X)$, and the corresponding unitary representation of $\mathrm{Diff}_0(X)$ in the $L^2$-space of the Poisson measure was studied in [38], see also [10]. Developing analysis associated with this representation, one naturally arrives at a differential structure on the configuration space $\Gamma(X)$, and derives a Laplace operator on $\Gamma(X)$, see [1]. In fact, one gets a certain lifting of the differential structure of the manifold $X$ to the configuration space $\Gamma(X)$. Hereby the Riemannian volume on $X$ is lifted to the Poisson measure on $\Gamma(X)$, and the Laplace–Beltrami operator on $X$, generated by the Dirichlet integral with respect to the Riemannian volume, is lifted to the generator of the Dirichlet form of the Poisson measure. The associated diffusion can be described as a Markov process on $\Gamma(X)$ in which movement of each point of configuration is a Brownian motion in $X$, independent of the other points of the configuration, see [16,30,31].

Let $C_0(X \to \mathbb{R}_+)$ denote the multiplicative group of continuous functions on $X$ with values in $\mathbb{R}_+ := (0, \infty)$ that are equal to one outside a compact set. (Analogously, we could have considered $C_0(X)$, the additive group of real-valued continuous functions on $X$ with compact support.) The group of diffeomorphisms, $\mathrm{Diff}_0(X)$, naturally acts on $X$, hence on $C_0(X \to \mathbb{R}_+)$. In this paper, we will consider the group

$$\mathfrak{G} = \mathrm{Diff}_0(X) \leftthreetimes C_0(X \to \mathbb{R}_+),$$