Journal of Functional Analysis 269 (2015) 2995–3041

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

K-theory for the tame C*-algebra of a separated graph [☆]

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ARTICLE INFO

Article history:
Received 9 April 2015
Accepted 12 May 2015
Available online 27 May 2015
Communicated by S. Vaes

MSC:
primary 46L35
secondary 46L80

Keywords:
Separated graph
Refinement monoid
K-theory
Partial action

ABSTRACT

A *separated graph* is a pair (E, C) consisting of a directed graph E and a set $C = \bigsqcup_{v \in E^0} C_v$, where each C_v is a partition of the set of edges whose terminal vertex is v. Given a separated graph (E, C), such that all the sets $X \in C$ are finite, the K-theory of the graph C*-algebra $C^*(E, C)$ is known to be determined by the kernel and the cokernel of a certain map, denoted by $1_C - A_{(E,C)}$, from $\mathbb{Z}^{(C)}$ to $\mathbb{Z}^{(E^0)}$. In this paper, we compute the K-theory of the *tame* graph C*-algebra $\mathcal{O}(E, C)$ associated to (E, C), which has been recently introduced by the authors. Letting π denote the natural surjective homomorphism from $C^*(E, C)$ onto $\mathcal{O}(E, C)$, we show that $K_1(\pi)$ is a group isomorphism, and that $K_0(\pi)$ is a split monomorphism, whose cokernel is a torsion-free abelian group. We also prove that this cokernel is a free abelian group when the graph E is finite, and determine its generators in terms of a sequence of separated graphs $\{(E_n, C^n)\}_{n=1}^{\infty}$ naturally attached to (E, C). On the way to showing our main results, we obtain an explicit description of a connecting map arising in a six-term exact sequence computing the K-theory of an amalgamated free

☆ The first named author was partially supported by DGI MICIIN-FEDER MTM2011-28992-C02-01, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya (2009SGR 1389). The second named author was partially supported by CNPq (300362/2010-2).

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http://dx.doi.org/10.1016/j.jfa.2015.05.010

product, and we also exhibit an explicit isomorphism between $\ker(1_C - A_{(E,C)})$ and $K_1(C^*(E,C))$.

1. Introduction

A *separated graph* is a pair (E, C) consisting of a directed graph E and a set $C = \bigsqcup_{v \in E^0} C_v$, where each C_v is a partition of the set of edges whose terminal vertex is v. Their associated C*-algebras $C^*(E, C)$ [4,1] provide generalizations of the usual graph C*-algebras (see e.g. [15]) associated to directed graphs, although these algebras behave quite differently from the usual graph algebras because the range projections corresponding to different edges need not commute. One motivation for their introduction was to provide graph-algebraic models for the C*-algebras $U^{\mathrm{nc}}_{m,n}$ studied by L. Brown [7] and McClanahan [11–13]. Another motivation was to obtain graph C*-algebras whose structure of projections is as general as possible. The theory of [4] was mainly developed for *finitely separated graphs*, which are those separated graphs (E, C) such that all the sets $X \in C$ are finite.

Recall that a set S of partial isometries in a C*-algebra $\mathcal{A}$ is said to be *tame* [9, Proposition 5.4] if every element of $U = \langle S \cup S^* \rangle$, the multiplicative semigroup generated by $S \cup S^*$, is a partial isometry. As indicated above, a main difficulty in working with $C^*(E, C)$ is that, in general, the generating set of partial isometries of these algebras is not tame. This is not the case for the usual graph algebras, where it can be easily shown that the generating set of partial isometries is tame. In order to solve this problem, we introduced in [2] the *tame graph C^*-algebra* $\mathcal{O}(E, C)$ of a separated graph. Roughly, this algebra is defined by imposing to $C^*(E, C)$ the relations needed to transform the canonical generating set of partial isometries into a tame set of partial isometries (see Section 2 for the precise definitions).

For a finite bipartite separated graph (E, C), a dynamical interpretation of the C*-algebra $\mathcal{O}(E, C)$ was obtained in [2], and using this, a useful representation of $\mathcal{O}(E, C)$ as a *partial crossed product* of a commutative C*-algebra by a finitely generated free group was derived. This theory enabled the authors to solve [2, Section 7] an open problem on paradoxical decompositions in a topological setting, posed in [10] and [18]. It is worth mentioning here that the restriction to bipartite graphs in this theory is harmless, since by [2, Proposition 9.1], we can attach to every separated graph (E, C) a bipartite separated graph $(\tilde{E}, \tilde{C})$ in such a way that the respective (tame) graph C*-algebras are Morita-equivalent.

One of the main technical tools in [2] is the introduction, for each finite bipartite separated graph (E, C), of a sequence of finite bipartite separated graphs $\{(E_n, C^n)\}$ such that the graph C*-algebras $C^*(E_n, C^n)$ approximate the tame graph C*-algebra $\mathcal{O}(E, C)$, in the sense that $\mathcal{O}(E, C) \cong \varinjlim_n C^*(E_n, C^n)$, see [2, Section 5].